

TECHNIQUES OF CONSTRUCTIONS OF VARIATIONS OF MIXED HODGE STRUCTURES

HISASHI KASUYA

ABSTRACT. We give a way of constructing real variations of mixed Hodge structures over compact Kähler manifolds by using mixed Hodge structures on Sullivan's 1-minimal models of certain differential graded algebras associated with real variations of Hodge structures.

1. INTRODUCTION

A *real variation of mixed Hodge structure* (\mathbb{R} -VMHS) over a complex manifold M is $(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*)$ so that:

- (1) \mathbf{E} is a local system of finite-dimensional \mathbb{R} -vector spaces.
- (2) \mathbf{W}_* is an increasing filtration of the local system \mathbf{E} .
- (3) \mathbf{F}^* is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
- (4) The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
- (5) For any $k \in \mathbb{Z}$, the local system $Gr_k^{\mathbf{W}}(\mathbf{E})$ with the filtration induced by \mathbf{F}^* is an real variation of real variation of Hodge structure of weight k .

The purpose of this paper is to give a way of constructing \mathbb{R} -VMHSs over compact Kähler manifolds starting from real variations of Hodge structures.

1.1. Prototype. We first introduce our main results for the simplest case. We briefly review Sullivan's 1-minimal model (see [3], [10], [22] for more details). We fix a ground field \mathbb{K} of characteristic zero. A differential graded algebra (DGA) \mathcal{M}^* is *1-minimal* if \mathcal{M}^* is the increasing union of sub-DGAs

$$\mathbb{K} = \mathcal{M}^*(0) \subset \mathcal{M}^*(1) \subset \dots$$

such that $\mathcal{M}^*(k)$ is the exterior algebra $\bigwedge(\mathcal{V}_1 \oplus \dots \mathcal{V}_k)$ of the direct sum $\mathcal{V}_1 \oplus \dots \mathcal{V}_k$ of vector spaces $\mathcal{V}_1, \dots, \mathcal{V}_k$ with $d\mathcal{V}_k \subset \mathcal{M}^2(k-1) = \bigwedge^2(\mathcal{V}_1 \oplus \dots \mathcal{V}_{k-1})$ for each $k \geq 1$ where the degree of any element in each \mathcal{V}_i is of degree 1. On the dual space of \mathcal{M}^1 , the dual map of the differential $d : \mathcal{M}^1 \rightarrow \mathcal{M}^1 \wedge \mathcal{M}^1$ is a Lie bracket. Such Lie algebra is called the *dual Lie algebra* of \mathcal{M}^* . A *1-minimal model* of a DGA A^* is a 1-minimal DGA \mathcal{M}^* with a morphism $\phi : \mathcal{M}^* \rightarrow A^*$ which induces isomorphisms on 0th and first cohomologies and an injection on second cohomology.

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For any cohomologically connected DGA A^* , a 1-minimal model \mathcal{M}^* of A^* exists and unique up to isomorphisms of DGAs where cohomologically connected means $H^0(A^*) \cong \mathbb{K}$.

Let M be a compact complex manifold with a Kähler metric g . Then, by Morgan's work ([16]), there exists an \mathbb{R} -mixed Hodge structure (W_*, F^*) on the 1-minimal model \mathcal{M}^* of the de Rham complex $A^*(M)$ of M . On the dual Lie algebra \mathfrak{u} of \mathcal{M}^* with the dual \mathbb{R} -mixed Hodge structure, we define:

Definition 1.1.1. A *mixed Hodge \mathfrak{u} -representation* is (V, W_*, F^*, Ω, b) so that:

- V is a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*)
- $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ is a representation which is compatible with the weight filtrations W_* .
- b is an automorphism of a \mathbb{C} -vector space $V \otimes \mathbb{C}$ which preserves the weight filtration W_* such that $b^{-1}\Omega b : \mathfrak{u}_{\mathbb{C}} \rightarrow \text{End}(V_{\mathbb{C}})$ is compatible with the Hodge filtrations F^* and the conjugate filtrations $\overline{F^*}$.

We notice that Ω is regarded as $\Omega \in \mathcal{M}^1 \otimes \text{End}(V)$ satisfying the Maurer-Cartan equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$. Thus, $\Omega_{\phi} = \phi(\Omega) \in A^*(M) \otimes \text{End}(V)$ gives a flat connection on the trivial \mathcal{C}^{∞} -vector bundle $M \times V$.

We state a prototype of the main result.

Theorem(Prototype). *We can take canonical maps $\phi : \mathcal{M}^* \rightarrow A^*(M)$ and $\varphi' : \mathcal{M}_{\mathbb{C}}^* \rightarrow A^*(M) \otimes \mathbb{C}$ which induce isomorphisms on 0th and first cohomologies and injections on second cohomology and define a special mixed Hodge structure (W_*, F^*) on the 1-minimal model \mathcal{M}^* of $A^*(M)$ so that for any mixed Hodge \mathfrak{u} -representation $\mathfrak{V} = (V, W_*, F^*, \Omega, b)$, we can construct an \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ satisfying the following conditions:*

- (1) $\mathbf{E}_{\mathfrak{V}}$ is the trivial \mathcal{C}^{∞} -vector bundle $M \times V$ with the flat connection $d + \Omega_{\phi}$ where $\Omega_{\phi} = \phi(\Omega) \in A^*(M) \otimes \text{End}(V)$.
- (2) $\mathbf{W}_{\mathfrak{V}*}$ is the filtration of the vector bundle $M \times V$ induced by the weight filtration W_* on V .
- (3) For some gauge transformation a of the vector bundle $M \times V$, we have $\mathbf{F}_{\mathfrak{V}}^* = ab\mathbf{F}^*$ such that \mathbf{F}^* is the filtration of the vector bundle $M \times V$ induced by the Hodge filtration F^* on V .
- (4) $\varphi'(\Omega) = a^{-1}da + a^{-1}\Omega_{\phi}a$.

Remark 1.1.2. \mathbb{R} -VMHSs $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ in this theorem are unipotent in the sense of Hain-Zucker [12]. In [12], by using the mixed-Hodge structure on the fundamental group derived from iterated integrals, Hain and Zucker constructed unipotent \mathbb{R} -VMHSs associated with unipotent mixed Hodge representations of the fundamental group. Our construction is very similar to Hain-Zucker's construction since Morgan's mixed Hodge structure can be regarded as a mixed Hodge structure on the "tensor product" of the fundamental group and the field \mathbb{R} by Sullivan's de Rham homotopy theory (see [16]). But they are different since Morgan's mixed Hodge structure on the 1-minimal model is different from the mixed-Hodge structure on the fundamental group derived from iterated integrals (see [17, Section 8 and 9]). In fact, for our

construction, we do not use "based point" unlike Hain-Zucker's construction. Our construction is depend on the choice of a Kähler metric. The maps $\phi : \mathcal{M}^* \rightarrow A^*(M)$ and $\varphi' : \mathcal{M}_{\mathbb{C}}^* \rightarrow A^*(M) \otimes \mathbb{C}$ canonically determined by a Kähler structure. An advantage of our construction is that we obtain an explicit globally defined connection forms $\Omega_{\phi} \in A^*(M) \otimes \text{End}(V)$ and $b^{-1}\varphi'(\Omega)b \in A^*(M) \otimes \text{End}(V \otimes \mathbb{C})$.

1.2. Main construction. In this paper we give an extended version of Theorem(Prototype) for obtaining non-unipotent \mathbb{R} -VMHSs. Let M be a compact Kähler manifold and $\rho : \pi_1(M, x) \rightarrow GL(V_0)$ be a real valued representation. Consider the real local system $\mathbf{E}_0 = (\tilde{M} \times V_0)/\pi_1(M, x)$ where \tilde{M} is the universal covering of M . We assume that \mathbf{E}_0 admits a \mathbb{R} -VHS $(\mathbf{E}_0, \mathbf{F}^*)$ with a polarization \mathbf{S} . Consider the bilinear form $\mathbf{S}_x : V_0 \times V_0 \rightarrow \mathbb{R}$. Then we have $\rho(\pi_1(M, x)) \subset T = \text{Aut}(V_0, \mathbf{S}_x)$. We assume that $\rho(\pi_1(M, x))$ is Zariski-dense in T .

In this assumption, we will set up the main construction by the following way

- Corresponding to $\rho : \pi_1(M, x) \rightarrow T$, we consider the DGA $A^*(M, \mathcal{O}_{\rho})$ of differential forms on M with values in a certain local system equipped with the T -action defined by Deligne and Hain [11].
- We construct the canonical T -equivariant real 1-minimal model $\phi : \mathcal{M}^* \rightarrow A^*(M, \mathcal{O}_{\rho})$ and the canonical T -equivariant complex 1-minimal model $\varphi : \mathcal{N}^* \rightarrow A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$.
- We construct an \mathbb{R} -mixed Hodge structure on \mathcal{M}^* by using $\varphi : \mathcal{N}^* \rightarrow A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ which relates to the T -action.
- For the \mathbb{R} -mixed Hodge structure on the dual Lie algebra \mathfrak{u} of \mathcal{M}^* , we define a *mixed Hodge (T, \mathfrak{u}) -module* (V, W_*, F^*, Ω, b) as a mixed Hodge \mathfrak{u} -representation as in Definition 1.1.1 with some conditions on T -actions.
- For any mixed Hodge (T, \mathfrak{u}) -representation $\mathfrak{V} = (V, W_*, F^*, \Omega, b)$, we construct an \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ satisfying similar conditions as in Theorem(Prototype).

Obtained \mathbb{R} -VMHSs $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ can be non-unipotent. For each irreducible representation V_{α} of T , we have a \mathbb{R} -VHS on $\mathbf{E}_{\alpha} = (\tilde{M} \times V_{\alpha})/\pi_1(M, x)$. Such variations appear in $Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathfrak{V}})$. We notice that our construction is closely related to Eyssidieux-Simpson's construction in [5]. By our construction, we can construct \mathbb{R} -VMHSs which are very similar to Eyssidieux-Simpson's VMHSs.

1.3. Arrangement of the paper. In Section 2–5, we will give basics of the main objects of this paper. The main part of this paper is Section 6–9. In Section 6, we will give details of constructions of canonical 1-minimal models and mixed Hodge structures. In Section 7, we will give the definition of mixed Hodge (T, \mathfrak{u}) -representations $\mathfrak{V} = (V, W_*, F^*, \Omega, b)$ and details of constructions of \mathbb{R} -VMHSs $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$. In Section 8 and Section 9, we will give techniques of producing mixed Hodge (T, \mathfrak{u}) -modules. When the weight of \mathbb{R} -VHS \mathbf{E}_0 is 0, in section 9, inspired by Eyssidieux-Simpson's work in [5], we will give \mathbb{R} -VMHSs starting from any T -module, by using the deformation theory of differential graded Lie algebras.

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2. VARIATIONS OF MIXED HODGE STRUCTURES

2.1. Mixed Hodge structures.

Definition 2.1.1. An \mathbb{R} -Hodge structure of weight n on a \mathbb{R} -vector space V is a bigrading

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

on the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that

$$\overline{V^{p,q}} = V^{q,p},$$

or equivalently, a finite decreasing filtration F^* on $V_{\mathbb{C}}$ such that

$$F^p(V_{\mathbb{C}}) \oplus \overline{F^{n+1-p}(V_{\mathbb{C}})} = V_{\mathbb{C}}$$

for each p .

A \mathbb{R} -Hodge structure of weight n corresponds to a rational representation $h : U(1) \rightarrow GL(V)$. This correspondence is given by

$$V^{p,q} = \{v \in V_{\mathbb{C}} | h(t)v = t^{n-2q}v\}$$

for $p + q = n$.

Definition 2.1.2. A polarization of an \mathbb{R} -Hodge structure of weight n is a $(-1)^n$ -symmetric bilinear form $S : V \times V \rightarrow \mathbb{R}$ so that:

- (1) The decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ is orthogonal for the sesquilinear form $S : V_{\mathbb{C}} \times \overline{V_{\mathbb{C}}} \rightarrow \mathbb{C}$.
- (2) $h : V_{\mathbb{C}} \times V_{\mathbb{C}} \rightarrow \mathbb{C}$ defined as $h(u, v) = S(Cu, \bar{v})$ is a positive-definite hermitian form where C is the Weil operator $C|_{V^{p,q}} = (\sqrt{-1})^{p-q}$.

Suppose V is finite-dimensional. Consider the homomorphism $h : U(1) \rightarrow GL(V)$ associated with an \mathbb{R} -Hodge structure of weight n . Then for a polarization S , we have $h(U(1)) \subset \text{Aut}(V, S)$.

Definition 2.1.3. An \mathbb{R} -mixed Hodge structure on an \mathbb{R} -vector space V is a pair (W_*, F^*) such that:

- (1) W_* is an increasing filtration,
- (2) F^* is a decreasing filtration on $V_{\mathbb{C}}$ such that the filtration on $Gr_n^W V_{\mathbb{C}}$ induced by F^* is an \mathbb{R} -Hodge structure of weight n .

We call W_* the weight filtration and F^* the Hodge filtration.

Proposition 2.1.4 ([2],[1],[16]). *Let (W_*, F^*) be an \mathbb{R} -mixed Hodge structure on an \mathbb{R} -vector space V . Define $V^{p,q} = R^{p,q} \cap L^{p,q}$ where*

$$R^{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap F^p(V_{\mathbb{C}})$$

and

$$L^{p,q} = W^{p+q}(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})} + \sum_{i \geq 2} W_{p+q-i}(V_{\mathbb{C}}) \cap \overline{F^{q-i+1}(V_{\mathbb{C}})}.$$

Then we have the bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ such that

$$\begin{aligned} \overline{V^{p,q}} &= V^{q,p} \quad \text{mod} \quad \bigoplus_{r+s < p+q} V^{r,s}, \\ W_i(V_{\mathbb{C}}) &= \bigoplus_{p+q \leq i} V^{p,q} \quad \text{and} \quad F^i(V_{\mathbb{C}}) = \bigoplus_{p \geq i} V^{p,q}. \end{aligned}$$

The bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is called the bigrading of a \mathbb{R} -mixed Hodge structure (W_*, F^*) . We have the converse statement.

Proposition 2.1.5 ([1],[16]). *Let V be an \mathbb{R} -vector space. We suppose that we have a bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ such that $\bigoplus_{p+q \geq n} V^{p,q}$ is an \mathbb{R} -subspace and*

$$\overline{V^{p,q}} = V^{q,p} \quad \text{mod} \quad \bigoplus_{r+s < p+q} V^{r,s}.$$

Then the filtrations W and F such that $W_i(V_{\mathbb{C}}) = \bigoplus_{p+q \leq i} V^{p,q}$ and $F^i(V_{\mathbb{C}}) = \bigoplus_{p \geq i} V^{p,q}$ give an \mathbb{R} -mixed Hodge structure on V .

Indeed, these two proposition gives an equivalence of categories between \mathbb{R} -mixed Hodge structures and bigrading as in Proposition 2.1.5 see [1].

2.2. Variations of hodge structures. Let M be a complex manifold.

Definition 2.2.1. A real variation of Hodge structure (\mathbb{R} -VHS) of weight $n \in \mathbb{Z}$ over M is a pair $(\mathbf{E}, \mathbf{F}^*)$ so that:

- (1) \mathbf{E} is a local system of finite-dimensional \mathbb{R} -vector spaces.
- (2) \mathbf{F}^* is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
- (3) The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
- (4) For any $x \in M$, (E_x, F_x^*) is a \mathbb{R} -Hodge structure of weight n .

For an \mathbb{R} -VHS of weight n , we have the decomposition $\mathbf{E}_{\mathbb{C}} = \bigoplus_{p+q=n} \mathbf{E}^{p,q}$ of \mathcal{C}^{∞} vector bundles so that $\mathbf{F}^r = \bigoplus_{p \geq r} \mathbf{E}^{p,q}$. By the Griffiths transversality, the differential D on $A^*(M, \mathbf{E}_{\mathbb{C}})$ decomposes $D = \partial + \bar{\partial} + \bar{\theta} + \theta$ so that:

$$\begin{aligned} \partial &: A^{a,b}(\mathbf{E}^{c,d}) \rightarrow A^{a+1,b}(\mathbf{E}^{c,d}), \\ \bar{\partial} &: A^{a,b}(\mathbf{E}^{c,d}) \rightarrow A^{a,b+1}(\mathbf{E}^{c,d}), \\ \theta &: A^{a,b}(\mathbf{E}^{c,d}) \rightarrow A^{a+1,b}(\mathbf{E}^{c-1,d+1}) \end{aligned}$$

and

$$\bar{\theta} : A^{a,b}(\mathbf{E}^{c,d}) \rightarrow A^{a,b+1}(\mathbf{E}^{c+1,d-1}).$$

We define

$$A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q} = \bigoplus_{a+c=P, b+d=Q} A^{a,b}(\mathbf{E}^{c,d}),$$

$D' = \partial + \bar{\theta}$ and $D'' = \bar{\partial} + \theta$. Then we have the double complex

$$(A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q}, D', D'')$$

as similar to the usual Dolbeault complex.

Definition 2.2.2. A polarization of an \mathbb{R} -VHS is a non-degenerate pairing $\mathbf{S} : \mathbf{E} \times \mathbf{E} \rightarrow \mathbb{R}$ so that for any $x \in M$ \mathbf{S}_x is a polarization of the \mathbb{R} -hodge structure (E_x, F_x^*) .

2.3. Variations of mixed Hodge structures.

Definition 2.3.1. A real variation of mixed Hodge structure (\mathbb{R} -VMHS) over M is $(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*)$ so that:

- (1) \mathbf{E} is a local system of finite-dimensional \mathbb{R} -vector spaces.
- (2) \mathbf{W}_* is an increasing filtration of the local system \mathbf{E} .
- (3) \mathbf{F}^* is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
- (4) The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
- (5) For any $k \in \mathbb{Z}$, the local system $Gr_k^{\mathbf{W}}(\mathbf{E})$ with the filtration induced by \mathbf{F}^* is an \mathbb{R} -VHS of weight k .

Example 2.3.2. We introduce essentially trivial cases:

- Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure. Regarding V as the trivial vector bundle over M , V is an \mathbb{R} -VMHS.
- Let $\mathbf{E}_1, \dots, \mathbf{E}_k$ be \mathbb{R} -VHSs. Then the direct sum $\mathbf{E}_1 \oplus \dots \oplus \mathbf{E}_k$ is an \mathbb{R} -VMHS.
- Let V_1, \dots, V_k be finite-dimensional \mathbb{R} -vector spaces with \mathbb{R} -mixed Hodge structures and $\mathbf{E}_1, \dots, \mathbf{E}_k$ be \mathbb{R} -VHSs. Then $\bigoplus V_i \otimes \mathbf{E}_i$ is \mathbb{R} -VMHS.

3. REPRESENTATIONS OF REDUCTIVE ALGEBRAIC GROUPS

3.1. Coordinate rings. Let T be a reductive algebraic group over \mathbb{R} and $\mathbb{R}[T]$ the coordinate ring of T . Let $\{V_{\alpha}\}$ be the set of isomorphism classes of irreducible representations of T . Then as an (T, T) -module, we have an isomorphism $\Theta : \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$ of T sending $f \otimes v \in V_{\alpha}^* \otimes V_{\alpha}$ to the function $T \ni t \rightarrow f(tv) \in \mathbb{R}$ (see [11, Section 3] or [23, Theorem 27.3.9]). Hence we have the algebra structure on $\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}$.

The multiplication on $\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}$ is given by decompositions of tensor products as the following way. For irreducible representations V_{α}, V_{β} , take an irreducible decomposition $V_{\alpha} \otimes V_{\beta} = \bigoplus_i V_{\gamma_i}$. For $f \otimes v \in V_{\alpha}^* \otimes V_{\alpha}$ and $g \otimes w \in V_{\beta}^* \otimes V_{\beta}$, for this decomposition, we take $f \otimes g = \sum h_i$ and $v \otimes w = \sum u_i$ so that $h_i \in V_{\gamma_i}^*$ and $u_i \in V_{\gamma_i}$. Then we have

$$\begin{aligned} (\Theta(f \otimes v)\Theta(g \otimes w))(t) &= f(tv)g(tw) = (f \otimes g)(t(v \otimes w)) \\ &= \left(\sum_i h_i \right) \left(\sum_j tu_j \right) = \sum_i h_i(tu_i) = \Theta \left(\sum h_i \otimes u_i \right) (t). \end{aligned}$$

Thus we have

$$(f \otimes v) \cdot (g \otimes w) = \sum h_i \otimes u_i.$$

Example 3.1.1. Let $T = SL_2(\mathbb{R})$ and V be the standard representation on \mathbb{R}^2 . We have

$$\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} = \bigoplus_k S^k V^* \otimes S^k V$$

Then $V \otimes V = \wedge^2 V \oplus S^2 V$. For $f \otimes v \in V^* \otimes V$ and $g \otimes w \in V^* \otimes V$

$$\begin{aligned} (f \otimes v) \cdot (g \otimes w) \\ = (f \wedge g) \otimes (v \wedge w) + (f \times g) \otimes (g \times w) \in \wedge^2 V^* \otimes \wedge^2 V \oplus S^2 V^* \otimes S^2 V \end{aligned}$$

where $\wedge^2 V = \mathbb{R}$.

3.2. Automorphism groups of polarizations. Let V be a \mathbb{R} -Hodge structure of weight n with a polarization S . Take the automorphism group $T = \text{Aut}(V, S)$. Consider T as an algebraic group over \mathbb{R} . We have $T(\mathbb{C}) = Sp_{2m}(\mathbb{C})$ when the weight n is odd or $T(\mathbb{C}) = O(m, \mathbb{C})$ when n is even. It is known that any irreducible representation V_{α} is given by

$$\mathbb{S}_{\lambda} V \cap V^{[d]}$$

so that \mathbb{S}_{λ} is the Schur functor associated with a partition λ of d and $V^{[d]}$ is the intersection of the kernels of all contractions $V^{\otimes d} \rightarrow V^{\otimes(d-2)}$ see [6]. Moreover, for certain set of partitions of numbers Λ , we have a bijection

$$\Lambda \ni \gamma \mapsto \mathbb{S}_{\lambda} V \cap V^{[d]} \in \{V_{\alpha}\}$$

By this, V_{α} admits a \mathbb{R} -hodge structure which is induced by the \mathbb{R} -hodge structure on V . These Hodge structures are described by the following way. Consider the homomorphism $h : U(1) \rightarrow GL(V)$ associated with the \mathbb{R} -hodge structure on V . Then we have $h(U(1)) \subset T$. Thus, the \mathbb{R} -hodge structure on V_{α} is determined by the homomorphism $\alpha \circ h : U(1) \rightarrow GL(V_{\alpha})$. Suppose $n = 0$. Then the Hodge structures on $\{V_{\alpha}\}$ are weight 0 and by this description, each irreducible decomposition of a tensor product $V_{\alpha} \otimes V_{\beta} = \bigoplus_i V_{\gamma_i}$ is a direct sum of \mathbb{R} -Hodge structures. Define the \mathbb{R} -Hodge structure on $\mathbb{R}[T] \cong \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}$ so that

$$\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \otimes \mathbb{C} = \bigoplus_{p+q=0} \bigoplus_{\alpha} (V_{\alpha}^*)^{p,q} \otimes V_{\alpha} \otimes \mathbb{C}.$$

Since \mathbb{R} -Hodge structures on V_{α} are compatible with irreducible decompositions of tensor products, the multiplication on $\mathbb{R}[T] \cong \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}$ is a morphism of \mathbb{R} -Hodge structure.

Let (V_1, S_1) and (V_2, S_2) be polarized \mathbb{R} -Hodge structures of same weight such that $\dim V_1 = \dim V_2$. For the decompositions $V_i \otimes \mathbb{C} = \bigoplus V_i^{p,q}$, we assume that $\dim V_1^{p,q} = \dim V_2^{p,q}$ for each (p, q) . Take $T_1 = \text{Aut}(V_1, S_1)$ and $T_2 = \text{Aut}(V_2, S_2)$. Then we have isomorphisms $T_1 \cong T$ and $T_2 \cong T$ for some orthogonal or symplectic group T . For an irreducible representation V_{α} of T , by these isomorphisms we regard V_{α}^* and V_{α} as a T_1 -module and a T_2 -module respectively. Consider the $T_1 \times T_2$ -module $V_{\alpha}^* \otimes V_{\alpha}$. Take a presentation $V_{\alpha} = \mathbb{S}_{\lambda} V \cap V^{[d]}$. Then V_{α}^* admits an

\mathbb{R} -Hodge structure induced by V_1 and V_α admits \mathbb{R} -Hodge structure induced by V_2 and so $V_\alpha^* \otimes V_\alpha$ admits an \mathbb{R} -Hodge structure of weight 0. By the similar argument as above, defining the \mathbb{R} -Hodge structure on $\mathbb{R}[T] \cong \bigoplus_\alpha V_\alpha^* \otimes V_\alpha$ so that

$$\bigoplus_\alpha V_\alpha^* \otimes V_\alpha \otimes \mathbb{C} = \bigoplus_{p+q=0} \bigoplus_\alpha \bigoplus_{s+u=p, t+v=q} (V_\alpha^*)^{s,t} \otimes V_\alpha^{u,v},$$

the multiplication on $\mathbb{R}[T] \cong \bigoplus_\alpha V_\alpha^* \otimes V_\alpha$ is a morphism of \mathbb{R} -Hodge structure.

4. DGA $A^*(M, \mathcal{O}_\rho)$

Let M be a \mathcal{C}^∞ -manifold, T a reductive algebraic group over \mathbb{R} and $\rho : \pi_1(M, x) \rightarrow T$ be a real valued representation. Assume that $\rho(\pi_1(M, x))$ is Zariski-dense in T . Let $\{V_\alpha\}$ be the set of isomorphism classes of irreducible representations of T . Consider the local systems $\mathbf{E}_\alpha = (\tilde{M} \times V_\alpha)/\pi_1(M, x)$. Denote by $A^*(M, \mathbf{E}_\alpha)$ the space of \mathbf{E}_α -valued \mathcal{C}^∞ -differential forms. Consider the cochain complex

$$A^*(M, \mathcal{O}_\rho) = \bigoplus_\alpha A^*(M, \mathbf{E}_\alpha^*) \otimes V_\alpha$$

with the differential $D = \bigoplus_\alpha D_\alpha$. Then by the wedge product and the multiplication on $\bigoplus_\alpha V_\alpha^* \otimes V_\alpha \cong \mathbb{R}[T]$, $(A(M, \mathcal{O}_\rho), D)$ is a cohomologically connected DGA with the T -action.

Let V a finite-dimensional T -module. We consider the algebra $A^*(M, \mathcal{O}_\rho) \otimes \text{End}(V)$ with the T -action. Then, by the isomorphism $(V \otimes \mathbb{R}[T])^T \cong V$, the space $(A^*(M, \mathcal{O}_\rho) \otimes \text{End}(V))^T$ is identified with $A^*(M, \text{End}(\mathbf{E}))$ where \mathbf{E} is the local system V associated with $\rho : \pi_1(M, x) \rightarrow T$ and the T -module structure on V . Hence, an element $\Omega \in (A^1(M, \mathcal{O}_\rho) \otimes \text{End}(V))^T$ satisfying the Maurer-Cartan equation $D\Omega + \frac{1}{2}[\Omega, \Omega] = 0$ gives the deformed flat connection $D + \Omega$ on the \mathcal{C}^∞ vector bundle \mathbf{E} .

5. HODGE THEORY ON COMPACT KÄHLER MANIFOLDS

Let M be a compact complex manifold. We assume that M admits a Kähler metric g . Let $(\mathbf{E}, \mathbf{F}^*)$ be a \mathbb{R} -VHS of weight n over M with a polarization \mathbf{S} . We consider the double complex $(A^*(M, \mathbf{E}_\mathbb{C})^{P,Q}, D', D'')$ as in SubSection 2.2. We define the differential operator $D^c = \sqrt{-1}(D'' - D')$ on the real valued differential forms $A^*(M, E)$. By the Hermitian metric on $E_\mathbb{C}$ associated with the polarization \mathbf{S} and the Kähler metric g , we define the adjoints D^* , $(D')^*$, $(D'')^*$ and $(D^c)^*$ of differential operators. For the Kähler form ω associated with g , we consider the adjoint operator Λ of the Lefschetz operator $A^*(M, \mathbf{E}) \ni \alpha \mapsto \omega \wedge \alpha \in A^{*+2}(M, E)$. Similarly to the usual Kähler identity, we have

$$[\Lambda, D] = -(D^c)^*$$

and this equation gives

$$\Delta_D = 2\Delta_{D'} = 2\Delta_{D''}$$

where Δ_D , $\Delta_{D'}$ and $\Delta_{D''}$ are the Laplacian operators (see [24]). Denote by

$$\mathcal{H}^r(M, \mathbf{E}) = \ker(\Delta_D)|_{A^r(M, \mathbf{E})} \quad \text{and} \quad \mathcal{H}^{P,Q}(M, \mathbf{E}_\mathbb{C}) = \ker(\Delta_{D''})|_{(A^*(M, \mathbf{E}_\mathbb{C}))^{P,Q}}.$$

Then we have the Hodge decomposition

$$\mathcal{H}^r(M, \mathbf{E}_{\mathbb{C}}) = \bigoplus_{P+Q=n+r} \mathcal{H}^{P,Q}(M, \mathbf{E}_{\mathbb{C}}).$$

Since Λ is a map of degree -2 , by the Kähler identity, we have the following useful equations

$$\mathcal{H}^1(M, \mathbf{E}) = \ker D|_{A^1(M, \mathbf{E})} \cap \ker D^c|_{A^1(M, \mathbf{E})}$$

and

$$\mathcal{H}^{P,Q}(M, \mathbf{E}_{\mathbb{C}}) = \ker D'|_{A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q}} \cap \ker D''|_{A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q}}$$

for $P + Q = 1 + n$.

By the same argument as [3, Section 5], we have the following DD^c -Lemma and $D'D''$ -Lemma.

Theorem 5.0.1. **(DD^c -Lemma):** *On $A^*(M, \mathbf{E})$,*

$$\operatorname{im} D \cap \ker D^c = \ker D \cap \operatorname{im} D^c = \operatorname{im} DD^c.$$

Moreover, there exists a linear map $F_g : \operatorname{im} D \cap \ker D^c \rightarrow A^(M, \mathbf{E}_{\mathbb{C}})^{P-1, Q-1}$ so that $\alpha = DD^c F_g \alpha$ for $\alpha \in \operatorname{im} D \cap \ker D^c$.*

($D'D''$ -Lemma): *On $A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q}$,*

$$\operatorname{im} D' \cap \ker D'' = \ker D' \cap \operatorname{im} D'' = \operatorname{im} D'D''.$$

Moreover, there exists a linear map $F'_g : \operatorname{im} D' \cap \ker D'' \rightarrow A^(M, \mathbf{E}_{\mathbb{C}})^{P-1, Q-1}$ so that $\alpha = D'D'' F'_g \alpha$ for $\alpha \in \operatorname{im} D' \cap \ker D''$.*

In fact, we can write $F_g = D^* G_D (D^c)^* G_{D^c}$ and $F'_g = (D')^* G_{D'} (D'')^* G_{D''}$ where $G_D, G_{D^c}, G_{D'}$ and $G_{D''}$ are the Green operators (see [3, Proof of (5.11)])

We consider the sub-complexes

$$\ker D^c \subset A^*(M, \mathbf{E}) \quad \text{and} \quad \ker D' \subset A^*(M, \mathbf{E}_{\mathbb{C}}).$$

The DD^c -Lemma and $D'D''$ -Lemma imply the following "formality" results (see [3, Section 6], also [8, Section 7]).

Corollary 5.0.2. • *The inclusions*

$$\ker D^c \subset A^*(M, \mathbf{E}) \quad \text{and} \quad \ker D' \subset A^*(M, \mathbf{E}_{\mathbb{C}})$$

induce cohomology isomorphisms.

• *The quotients*

$$\ker D^c \rightarrow H^*(A^*(M, \mathbf{E}), D^c) \quad \text{and} \quad \ker D' \rightarrow H^*(A^*(M, \mathbf{E}_{\mathbb{C}}), D')$$

induce cohomology isomorphisms.

• *We have isomorphisms*

$$H^*(A^*(M, \mathbf{E}), D) \cong H^*(A^*(M, \mathbf{E}), D^c)$$

and

$$H^*(A^*(M, \mathbf{E}_{\mathbb{C}}), D) \cong H^*(A^*(M, \mathbf{E}_{\mathbb{C}}), D').$$

6. 1-MINIMAL MODELS ON COMPACT KÄHLER MANIFOLDS

6.1. Assumptions. Let M be a compact Kähler manifold and $\rho : \pi_1(M, x) \rightarrow GL(V_0)$ be a real valued representation. Consider the real local system $\mathbf{E}_0 = (\tilde{M} \times V_0)/\pi_1(M, x)$ where \tilde{M} is the universal covering of M . We assume that \mathbf{E}_0 admits a \mathbb{R} -VHS $(\mathbf{E}_0, \mathbf{F}^*)$ of weight N_0 with a polarization \mathbf{S} . Consider the bilinear form $\mathbf{S}_x : V_0 \times V_0 \rightarrow \mathbb{R}$. Then we have $\rho(\pi_1(M, x)) \subset T = \text{Aut}(V_0, \mathbf{S}_x)$. We assume that $\rho(\pi_1(M, x))$ is Zariski-dense in T .

We will divide the following two cases

- The weight $N_0 = 0$, or
- The weight N_0 is odd.

We notice that any polarized \mathbb{R} -VHS of even weight can be shifted to the polarized \mathbb{R} -VHS of weight 0 and so we do not add any further geometric restriction.

The assumption $N_0 = 0$ is very helpful. In this case, we will obtain T -invariant mixed Hodge structures on 1-minimal models and we do not need the Hodge structure on the Lie algebra of T . On the other hand, in the case N_0 is odd, we need the Hodge structure on the Lie algebra of T .

6.2. Summary of this section. In this section, we will give Morgan's mixed Hodge structure on the 1-minimal model of the DGA $A^*(M, \mathcal{O}_\rho)$ by the following way:

- (1) We consider the DGA $A^*(M, \mathcal{O}_\rho)$ associated with the representation $\rho : \pi_1(M, x) \rightarrow T$. We take the structure of bidifferential bigraded algebra on $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ by using the variation on \mathbf{E}_0 . (Subsection 6.3)
- (2) We construct a T -equivariant "real" 1-minimal model $\phi : \mathcal{M}^* \rightarrow A^*(M, \mathcal{O}_\rho)$ with a grading $\mathcal{M}^* = \bigoplus \mathcal{M}_k^*$. (Subsection 6.4)
- (3) We construct a T -equivariant "complex" 1-minimal model $\varphi : \mathcal{N}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ with a bigrading $\mathcal{N}^* = \bigoplus (\mathcal{N}^*)^{P, Q}$. (Subsection 6.5)
- (4) We take a T -equivariant isomorphism $\mathcal{I} : \mathcal{M}_\mathbb{C}^* \rightarrow \mathcal{N}^*$ which is compatible with filtrations $W_k(\mathcal{M}^*) = \bigoplus_{i \leq k} \mathcal{M}_i^*$ and $W_k(\mathcal{N}^*) = \bigoplus_{P+Q \leq k} (\mathcal{N}^*)^{P, Q}$ and a T -equivariant homotopy $H : \mathcal{M}_\mathbb{C}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ from $\varphi \circ \mathcal{I}$ to ϕ . (Subsection 6.6)
- (5) By $F^r(\mathcal{M}_\mathbb{C}^*) = \mathcal{I}^{-1}(\bigoplus_{P \geq r} (\mathcal{N}^*)^{P, Q})$, we define a \mathbb{R} -mixed Hodge structure on \mathcal{M}^* . (Subsection 6.6)

Avoiding the arguments on T , we can find these constructions in [3] and [16] for the usual de Rham complex $A^*(M)$.

6.3. DGAs $A^*(M, \mathcal{O}_\rho)$ on compact Kähler manifolds. Let $\{V_\alpha\}$ be the set of isomorphism classes of irreducible representations of T . Consider the local systems $\mathbf{E}_\alpha = (\tilde{M} \times V_\alpha)/\pi_1(M, x)$. Then, for certain set Λ of partitions of numbers, we have a bijection

$$\Lambda \ni \lambda \mapsto S_\lambda \mathbf{E}_0 \cap \mathbf{E}^{[d]} \in \{\mathbf{E}_\alpha\} \quad (S_\lambda V_0 \cap V^{[d]} \in \{V_\alpha\}).$$

Hence, any \mathbf{E}_α (resp. V_α admits a polarized \mathbb{R} -VHS (resp \mathbb{R} -Hodge structure) induced by $(\mathbf{E}, \mathbf{F}^*)$ (resp. (V_0, \mathbf{F}_x^*)).

We consider the DGA $A^*(M, \mathcal{O}_\rho)$ as in Section 4. We define the bigrading on $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ as the following way.

Definition 6.3.1. • If $N_0 = 0$, then

$$A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q} = \bigoplus_{\alpha} A^*(M, \mathbf{E}_\alpha^* \otimes \mathbb{C})^{P,Q} \otimes V_\alpha \otimes \mathbb{C}.$$

• If N_0 is odd, then

$$A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q} = \bigoplus_{\alpha} \bigoplus_{S+U=P, T+V=Q} A^*(M, \mathbf{E}_\alpha^* \otimes \mathbb{C})^{S,T} \otimes (V_\alpha \otimes \mathbb{C})^{U,V}.$$

By the argument in Subsection 3.2, on each case, we can say that the product on $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ is compatible with the bigrading. Hence, we have the bidifferential bigraded algebra structure

$$(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}, D', D'').$$

If $N_0 = 0$, the T -action is compatible with this structure. If N_0 is odd, the T -action commutes with D' and D'' but T -action does not preserve the bigrading $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}$. In this case, for the Lie algebra \mathfrak{t} of T with the \mathbb{R} -Hodge structure induced by $(V_0, \mathbf{F}_x^*, \mathbf{S}_x)$, the \mathfrak{t} -action is compatible with the bigrading on $\mathfrak{t} \otimes \mathbb{C}$ and the bigrading $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}$. In both cases, we have

$$A^R(M, \mathcal{O}_\rho \otimes \mathbb{C}) = \bigoplus_{R=P+Q} A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}.$$

Example 6.3.2. Let M be a compact Riemann surface of genus $g \geq 2$. Then M is a compact quotient \mathbb{H}/Γ of the upper-half plane \mathbb{H} by a discrete subgroup Γ in $PSL_2(\mathbb{R})$. Take a lifting $\rho : \Gamma \rightarrow SL_2(\mathbb{R})$ of the embedding of Γ into $PSL_2(\mathbb{R})$. By the Borel density, $\rho(\Gamma)$ is Zariski-dense in $SL_2(\mathbb{R})$. Consider the local system $\mathbf{E}_0 = (\mathbb{H} \times \mathbb{R}^2)/\Gamma$. We regard \mathbb{H} as the classifying space of polarized \mathbb{R} -Hodge structures of weight 1 on \mathbb{R}^2 . Then, considering the identity map on \mathbb{H} as a period map, the local system \mathbf{E}_0 admits a polarized \mathbb{R} -VHS by taking certain decomposition $\mathbf{E}_0 \otimes \mathbb{C} = \mathbf{E}_0^{1,0} \oplus \mathbf{E}_0^{0,1}$. It is known that we can take $\mathbf{E}_0^{1,0} = K^{\frac{1}{2}}$ and $\mathbf{E}_0^{0,1} = K^{-\frac{1}{2}}$ where $K^{\frac{1}{2}}$ is a square-root of the canonical bundle K on M and $D'' = \bar{\partial} + \theta$ such that θ is $1 \in K \otimes \text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \cong \mathbb{C}$ (see [14], [21] and [7]).

We consider the DGA $A^*(M, \mathcal{O}_\rho)$. By Example 3.1.1, we have

$$A^*(M, \mathcal{O}_\rho) = \bigoplus_{k=0}^{\infty} A^*(M, S^k \mathbf{E}_0^*) \otimes S^k V_0$$

where $V_0 = (\mathbf{E}_0)_x$. By the multiplication on the coordinate ring $\mathbb{R}[SL_2(\mathbb{R})]$ (see Example 3.1.1), we have

$$(A^*(M, \mathbf{E}_0^*) \otimes V_0) \wedge (A^*(M, \mathbf{E}_0^*) \otimes V_0) \subset (A^*(M, \mathbb{R}) \otimes \mathbb{R}) \oplus (A^*(M, S^2 \mathbf{E}_0^*) \otimes S^2 V_0).$$

Denote $\mathbf{E}_0^{p,q} = (\mathbf{E}_0^{1,0})^{\otimes p} \otimes (\mathbf{E}_0^{0,1})^{\otimes q}$ and \cdot . Then we have

$$A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{0 \leq i, j \leq k} A^{*,*}(M, (\mathbf{E}_0^{i, k-i})^*) \otimes V_0^{j, k-j}.$$

For the bigrading $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P, Q}$ as the above argument, an element in

$$A^{a,b}(M, (\mathbf{E}_0^{i,k-1})^*) \otimes V_0^{j,k-j}$$

is of type $(a - i + j, b + i - j)$. By the multiplication on $\mathbb{R}[SL_2(\mathbb{R})]$, we have

$$\begin{aligned} & \left(A^{a,b}(M, (\mathbf{E}_0^{1,0})^*) \otimes V_0^{0,1} \right) \wedge \left(A^{c,d}(M, (\mathbf{E}_0^{0,1})^*) \otimes V_0^{1,0} \right) \\ & \subset \left(A^{a+c,b+d}(M, \mathbb{C}) \otimes \mathbb{C} \right) \oplus \left(A^{a+c,b+d}(M, (\mathbf{E}_0^{1,1})^*) \otimes V_0^{1,1} \right). \end{aligned}$$

6.4. The 1-minimal model associated with DD^c -Lemma. On the DGA $A^*(M, \mathcal{O}_\rho)$ with the T -action, we also consider another differential $D^c = \sqrt{-1}(D'' - D')$. Then, by Theorem 5.0.1 and the last subsection, we can say that on $A^*(M, \mathcal{O}_\rho)$, we have the equality

$$\text{im} D \cap \ker D^c = \ker D \cap \text{im} D^c = \text{im} DD^c$$

and there exist a T -equivariant linear map $F_g : \text{im} D \cap \ker D^c \rightarrow A^{*-2}(M, \mathcal{O}_\rho)$ so that $\alpha = DD^c F_g \alpha$ for $\alpha \in \text{im} D \cap \ker D^c$. By using this, we construct the DGAs $\mathcal{M}^*(n) = \bigwedge(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n)$ generated by elements of degree 1 and the homomorphisms $\phi_n : \mathcal{M}^*(n) \rightarrow \ker D^c \subset A^*(M, \mathcal{O}_\rho)$ as the following inductive way.

- $\mathcal{V}_1 = \ker D \cap \ker D^c \cap A^1(M, \mathcal{O}_\rho)$, the homomorphism $\phi_1 : \bigwedge \mathcal{V}_1 \rightarrow \ker D^c$ so that on \mathcal{V}_1 , ϕ_1 is the natural inclusion $\mathcal{V}_1 \hookrightarrow \ker D^c$.
- For the quotient map $q : \ker D \rightarrow H^*(\ker D^c)$,

$$\mathcal{V}_2 = \ker \left(\phi_1 \circ q : \bigwedge^2 \mathcal{V}_1 \rightarrow H^2(\ker D^c) \right)$$

Define the DGA $\mathcal{M}^*(2) = \bigwedge(\mathcal{V}_1 \oplus \mathcal{V}_2)$ with the differential d so that d is 0 on \mathcal{V}_1 and d on \mathcal{V}_2 is the natural inclusion $\mathcal{V}_2 \hookrightarrow \bigwedge^2 \mathcal{V}_1$. Define the homomorphism $\phi_2 : \mathcal{M}^*(2) \rightarrow \ker D^c$ which is an extension of ϕ_1 so that $\phi_2(v) = D^c F_g(\phi_1(dv))$.

- For $n \geq 2$, consider the DGA $\mathcal{M}^*(n) = \bigwedge(\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n)$ with the homomorphism $\phi_n : \mathcal{M}^*(n) \rightarrow \ker D^c$ we have constructed. We can say that $\phi_n(v) \in \ker D \cap \ker D^c$ for $v \in \mathcal{V}_1$ and as an inductive hypothesis we assume $\phi_n(v) \in \text{im} D^c$ for $v \in \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$.

Let

$$\mathcal{V}_{n+1} = \ker d|_{\sum_{i+j=n+1} \mathcal{V}_i \wedge \mathcal{V}_j}$$

. Define the extended DGA $\mathcal{M}^*(n+1) = \mathcal{M}^*(n) \otimes \bigwedge \mathcal{V}_{n+1}$ so that the differential d is defined on \mathcal{V}_{n+1} as the natural inclusion $\mathcal{V}_{n+1} \hookrightarrow \sum_{i+j=n+1} \mathcal{V}_i \wedge \mathcal{V}_j$. The homomorphism $\phi_{n+1} : \mathcal{M}^*(n+1) \rightarrow \ker D^c$ is defined by $\phi_{n+1}(v) = D^c F_g(\phi_n(dv))$ for $v \in \mathcal{V}_{n+1}$.

Let $\varinjlim \mathcal{M}^*(n) = \mathcal{M}^*$ and $\phi = \varinjlim \phi_n : \mathcal{M}^* \rightarrow \ker D^c \subset A^*(M, \mathcal{O}_\rho)$.

By the construction, each \mathcal{V}_i is a \hat{T} -module so that the map $\phi : \mathcal{M}^* \rightarrow A^*(M, \mathcal{O}_\rho)$ is T -equivariant. Our construction is in fact the construction of a 1-minimal model of $\ker D^c$ (see [10, Theorem 13.1]). Since the inclusion $\ker D^c \subset A^*(M, \mathcal{O}_\rho)$ induces a cohomology isomorphism, we can say that the map $\phi : \mathcal{M}^* \rightarrow A^*(M, \mathcal{O}_\rho)$ induces isomorphisms on 0-th and first cohomology and an injection on second cohomology.

Define the multiplicative grading \mathcal{M}_k^* so that elements in \mathcal{V}_k are of type k . Then, each \mathcal{M}_k^* is a T -module and we have $d\mathcal{M}_k^* \subset \mathcal{M}_k^*$. Define the increasing filtration W_* so that $W_n(\mathcal{M}^*) = \bigoplus_{k \leq n} \mathcal{M}_k^*$.

6.5. The 1-minimal model associated with $D'D''$ -Lemma. Consider the bi-differential bigraded algebra

$$(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}, D', D'').$$

Then, by Theorem 5.0.1, we can say that on $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}$,

$$\text{im} D' \cap \ker D'' = \ker D' \cap \text{im} D'' = \text{im} D' D''.$$

and there exist a T -equivariant linear map $F'_g : \text{im} D' \cap \ker D'' \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P-1, Q-1}$ so that $\alpha = D' D'' F'_g \alpha$ for $\alpha \in \text{im} D' \cap \ker D''$. By using this, we construct the DGAs $\mathcal{N}^*(n)$ generated by elements of degree 1 and the homomorphisms $\varphi_n : \mathcal{N}^*(n) \rightarrow \ker D' \subset A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ as the following inductive way.

- For $P, Q \in \mathbb{Z}$ with $P+Q=1$, let $\mathcal{V}^{P,Q} = \ker D' \cap \ker D'' \cap A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}$. Define the homomorphism $\varphi_1 : \bigwedge(\bigoplus_{P+Q=1} \mathcal{V}^{P,Q}) \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ so that on $\mathcal{V}^{P,Q}$, φ_1 is natural inclusion $\mathcal{V}^{P,Q} \hookrightarrow A^1(M, \mathcal{O}_\rho \otimes \mathbb{C})$.
- For the quotient map $q : \ker D' \cap \ker D'' \rightarrow H^*(\ker D')$, for $P, Q \in \mathbb{Z}$ with $P+Q=2$, define

$$\mathcal{V}^{P,Q} = \ker \left(\varphi_1 \circ q : \bigoplus_{S+T=P, U+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V} \rightarrow H^2(\ker D') \right).$$

Define the DGA $\mathcal{N}^*(2) = \bigwedge(\bigoplus_{1 \leq P+Q \leq 2} \mathcal{V}^{P,Q})$ with the differential d so that d is 0 on $\mathcal{V}^{P,Q}$ for $P+Q=1$ and d on $\mathcal{V}^{P,Q}$ is the natural inclusion $\mathcal{V}^{P,Q} \hookrightarrow \bigoplus_{S+T=P, U+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V}$. Define the homomorphism $\varphi_2 : \mathcal{N}^*(2) \rightarrow \ker D'$ which is an extension of φ_1 so that $\varphi_2(v) = D' F'_g(\varphi_1(dv))$ for $v \in \mathcal{V}^{P,Q}$ with $P+Q=2$.

- For $n \geq 2$, consider the DGA $\mathcal{N}^*(n) = \bigwedge(\bigoplus_{1 \leq P+Q \leq n} \mathcal{V}^{P,Q})$ with the homomorphism $\varphi_n : \mathcal{N}^*(n) \rightarrow A^*(M, \mathcal{O}_\rho)$ we have constructed. We can say that $\varphi_n(v) \in \ker D \cap \ker D^c$ for $v \in \mathcal{V}^{P,Q}$ with $P+Q=1$ and as an inductive hypothesis we assume $\varphi_n(v) \in \text{im} D'$ for $v \in \mathcal{V}^{P,Q}$ with $P+Q \geq 2$.

For $P+Q=n+1$, let

$$\mathcal{V}^{P,Q} = \ker d|_{\sum_{S+T=P, U+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V}}.$$

Define the extended DGA $\mathcal{N}^*(n+1) = \mathcal{N}^*(n) \otimes \bigwedge(\bigoplus_{P+Q=n+1} \mathcal{V}^{P,Q})$ so that the differential d is defined on $\mathcal{V}^{P,Q}$ with $P+Q=n+1$ as the natural inclusion $\mathcal{V}^{P,Q} \hookrightarrow \sum_{S+T=P, U+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V}$. The homomorphism $\varphi_{n+1} : \mathcal{N}^*(n+1) \rightarrow A^*(M, \mathcal{O}_\rho)$ is defined by $\varphi_{n+1}(v) = D' F'_g(\phi_n(dv))$ for $v \in \mathcal{V}^{P,Q}$ with $P+Q=n+1$.

Let $\varinjlim \mathcal{N}^*(n) = \mathcal{N}^*$ and $\varphi = \varinjlim \varphi_n : \mathcal{N}^* \rightarrow \ker D'' \subset A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$.

By the construction, each $\bigoplus_{P+Q=k} \mathcal{V}^{P,Q}$ is a T -module so that the map $\varphi : \mathcal{N}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ is T -equivariant. Our construction is in fact the construction of a

1-minimal model of $\ker D''$. Since the inclusion $\ker D'' \subset A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ induces a cohomology isomorphism, we can say that the map $\varphi : \mathcal{N}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ induces isomorphisms on 0-th and first cohomology and an injection on second cohomology.

Define the multiplicative bigrading $(\mathcal{N}^*)^{P,Q}$ so that elements in $\mathcal{V}_{P,Q}$ are of type (P, Q) . By the construction, each $(\mathcal{N}^*)^{P,Q}$ is a T -submodule and we have $d(\mathcal{N}^*)^{P,Q} \subset (\mathcal{N}^*)^{P,Q}$. Define the increasing filtration W_* so that $W_n(\mathcal{N}^*) = \bigoplus_{P+Q \leq n} (\mathcal{N}^*)^{P,Q}$.

If $N_0 = 0$, then each $\mathcal{V}_{P,Q}$ is a T -module and so the T -action on \mathcal{N}^* preserves the bigrading $(\mathcal{N}^*)^{P,Q}$.

If N_0 is odd, then for the Lie algebra \mathfrak{t} of T with the \mathbb{R} -Hodge structure $\mathfrak{t}_{\mathbb{C}} = \bigoplus_{p+q=0} \mathfrak{t}^{p,q}$ induced by (V, F_x^*, S_x) , the \mathfrak{t} -action satisfies $\mathfrak{t}^{p,q} \otimes \mathcal{V}^{P,Q} \rightarrow \mathcal{V}^{p+P, q+Q}$. Hence, in this case, the \mathfrak{t} -action $\mathfrak{t}_{\mathbb{C}} \otimes \mathcal{N}^* \rightarrow \mathcal{N}^*$ is compatible with the bigradings.

6.6. The MHS on the 1-minimal model. By Corollary 5.0.2, we have the quasi-isomorphism $\ker D^c \rightarrow H^*(M, \mathcal{O}_\rho)$. Thus \mathcal{M}^* as in Subsection 6.4 is the 1-minimal model of $H^*(M, \mathcal{O}_\rho)$. By the increasing filtration on $H^*(M, \mathcal{O}_\rho)$ associated with the degree and the decreasing filtration associated with the Hodge structure of each $H^n(M, \mathcal{O}_\rho \otimes \mathbb{C})$, the pair $(H^*(M, \mathcal{O}_\rho), H^n(M, \mathcal{O}_\rho \otimes \mathbb{C}))$ is a mixed hodge diagram in the sense of Morgan ([16]). Hence, by the Morgan's result in [16], the 1-minimal model \mathcal{M}^* admits an \mathbb{R} -mixed Hodge structure. We should remark that Morgan's mixed Hodge structure is not unique in general.

In this subsection, for our purpose, we will explain the construction of a special \mathbb{R} -mixed Hodge structure on \mathcal{M}^* precisely by using the complex 1-minimal model \mathcal{N}^* as in Subsection 6.5.

Proposition 6.6.1. *There exists a T -equivariant isomorphism $\mathcal{I} : \mathcal{M}_{\mathbb{C}}^* \rightarrow \mathcal{N}$ which is compatible with the increasing filtrations*

$$W_k(\mathcal{M}^*) = \bigoplus_{i \leq k} \mathcal{M}_i^* \quad \text{and} \quad W_k(\mathcal{N}^*) = \bigoplus_{P+Q \leq k} (\mathcal{N}^*)^{P,Q}$$

and T -equivariant homotopy $H : \mathcal{M}_{\mathbb{C}}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ from $\varphi \circ \mathcal{I}$ to ϕ where $[t, dt]$ means the tensor product of polynomials on t with the exterior algebra of dt (see e.g. [10, Chapter 11]).

Proof. In fact, for the proof, it is sufficient to trace the arguments in [16, (5.5)~(5.9), (7.3)~(7.5)]. We show by induction. We assume that there exists $\mathcal{I} : \mathcal{M}_{\mathbb{C}}^*(n) \rightarrow \mathcal{N}^*(n)$ so that there exists a homotopy $H : \mathcal{M}_{\mathbb{C}}^*(n) \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ from ϕ_n to $\varphi_n \circ \mathcal{I}$. For $v \in \mathcal{V}_{n+1}$, since we have

$$\phi_n(dv) - \varphi_n(\mathcal{I}(dv)) = D \int_0^1 H(dv),$$

$[\varphi(\mathcal{I}(dv))] = 0$ in $H^2(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}))$, we can take $a(v) \in W_{n+1}(\mathcal{N}^1)$ such that $da(v) = \mathcal{I}(dv)$. We notice that if we assume $a(v) \in \bigoplus_{P+Q \geq 2} \mathcal{V}^{P,Q}$, then $a(v)$ is unique. Consider

$$\phi(v) - \varphi(a(v)) - \int_0^1 H(dv).$$

Then it is closed and so we have a unique $a'(v) \in \bigoplus_{P+Q=1} \mathcal{V}^{P,Q}$ such that

$$[\varphi(a'(v))] = \left[\phi(v) - \varphi(a(v)) - \int_0^1 H(dv) \right]$$

in $H^1(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}))$. Now extend $\mathcal{I} : \mathcal{M}_\mathbb{C}^*(n+1) \rightarrow \mathcal{N}^*(n+1)$ so that $\mathcal{I}(v) = -a(v) - a'(v)$ choosing $a(v)$, $a'(v)$ for each $v \in \mathcal{V}_{n+1}$ such that \mathcal{I} on \mathcal{V}_{n+1} is linear. Then, we can extend $H : \mathcal{M}_\mathbb{C}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ so that for $v \in \mathcal{V}_{n+1}$

$$H(v) = \varphi(\mathcal{I}(v)) + \int_0^t H(dv) - D(b(v)t)$$

where $b(v)$ is an element in $A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ satisfying $D b(v) = \phi(v) - \varphi(a(v) + a'(v)) + \int_0^1 H(dv)$.

We can easily take \mathcal{I} and H as T -equivariant maps inductively. Decompose \mathcal{V}_n and $\mathcal{V}_{P,Q}$ as T -modules such that each block corresponds to a irreducible representation of T . Then, on the inductive process, we can extend \mathcal{I} and H block by block. \square

Remark 6.6.2. It is not obvious that we can take $\mathcal{I}(\mathcal{V}_n \otimes \mathbb{C}) \subset \bigoplus_{P+Q=n} \mathcal{V}^{P,Q}$ for $k \geq 3$. For $n = 1$, by the constructions, we can take $\mathcal{I} : \mathcal{V}_1 \otimes \mathbb{C} \cong \bigoplus_{P+Q=1} \mathcal{V}^{P,Q}$ so that $\phi_1 = \varphi_1 \circ \mathcal{I}$. For $n = 2$, we can take $\mathcal{I} : \mathcal{V}_2 \otimes \mathbb{C} \cong \bigoplus_{P+Q=2} \mathcal{V}^{P,Q}$ so that the homotopy $H : \mathcal{M}_\mathbb{C}^*(n) \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ is given by

$$H(v) = -2\sqrt{-1}D'\alpha + \sqrt{-1}D\alpha t + \sqrt{-1}\alpha dt$$

for $v \in \mathcal{V}_2$ satisfying $\phi(v) = D^c\alpha$.

Remark 6.6.3. In this proposition, I and H are not unique. But, in the construction given in the proof, ambiguity occurs only from the choices of $b(v)$. By using the Green operator of a Kähler metric, $b(v)$ is uniquely determined in each step. Hence, we can take canonical I and H determined by a Kähler metric.

By the map $\mathcal{I} : \mathcal{M}_\mathbb{C}^* \rightarrow \mathcal{N}$ constructed in this proposition. We obtain Morgan's mixed Hodge structure ([16]).

Theorem 6.6.4. *For an isomorphism $\mathcal{I} : \mathcal{M}_\mathbb{C}^* \rightarrow \mathcal{N}$ as in Proposition 6.6.1, taking the filtration $F^r(\mathcal{M}_\mathbb{C}^*) = \mathcal{I}^{-1}(\bigoplus_{P \geq r} (\mathcal{N}^*)^{P,Q})$, (\mathcal{M}^*, W, F) is an \mathbb{R} -mixed hodge structure which is compatible with the differential, the multiplication.*

If $N_0 = 0$, then the \mathbb{R} -mixed hodge structure is preserved by the T -action. If N_0 is odd, then for the Lie algebra \mathfrak{t} of T with the \mathbb{R} -Hodge structure induced by (V, F_x^, S_x) , the \mathfrak{t} -action $\mathfrak{t} \otimes \mathcal{M}^* \rightarrow \mathcal{M}^*$ is a morphism of \mathbb{R} -mixed Hodge structure.*

Proof. It is sufficient to show that for any n , $(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n, W_*, F^*)$ is an \mathbb{R} -mixed hodge structure. For $n = 1$, by $\mathcal{V}_1 = \ker D \cap \ker D^c$ and $\mathcal{V}^{P,Q} = \ker D' \cap \ker D'' \cap A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}$ for $P + Q = 1$, since we have

$$\ker D \cap \ker D^c = \mathcal{H}^1(M, \mathcal{O}_\rho)$$

and

$$\ker D' \cap \ker D'' \cap A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q} = \mathcal{H}^{P,Q}(M, \mathcal{O}_\rho \otimes \mathbb{C}),$$

the hodge decomposition

$$\mathcal{H}^1(M, \mathcal{O}_\rho) = \bigoplus_{P+Q=1} \mathcal{H}^{P,Q}(M, \mathcal{O}_\rho \otimes \mathbb{C}),$$

(\mathcal{V}_1, F^*) is an \mathbb{R} -hodge structure of weight 1.

For $n = 2$, as we see Remark 6.6.2, we have $\mathcal{I}(\mathcal{V}_2) \subset \bigoplus_{P+Q=2} \mathcal{V}_{P,Q}$. Hence, we show that (\mathcal{V}_2, F^*) is an \mathbb{R} -hodge structure of weight 2. This follows from $\mathcal{V}_2 = \ker(\phi \circ q : \bigwedge^2 \mathcal{V}_1 \rightarrow H^2(\ker D^c))$ since the cup product $H^1(M, \mathcal{O}_\rho) \otimes H^1(M, \mathcal{O}_\rho) \rightarrow H^2(M, \mathcal{O}_\rho)$ is a morphism of \mathbb{R} -hodge structure.

For $n \geq 2$, we use induction. Assume $(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n, W_*, F^*)$ is an \mathbb{R} -mixed hodge structure. Then we show that $(Gr_n^W(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n \oplus \mathcal{V}_{n+1}), F^*)$ is an \mathbb{R} -hodge structure of weight $n + 1$. This follows from

$$Gr_n^W(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n \oplus \mathcal{V}_{n+1}) = \ker(d|Gr_n^W \mathcal{M}^2 \rightarrow Gr_n^W \mathcal{M}^3)$$

since $\ker(d|Gr_{n+1}^W \mathcal{M}^2 \rightarrow Gr_{n+1}^W \mathcal{M}^3)$ is a morphism of \mathbb{R} -hodge structure by the inductive assumption and the induced filtration F^* on $Gr_n^W(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n \oplus \mathcal{V}_{n+1})$ is identified with the Hodge filtration of $\ker(d|Gr_{n+1}^W \mathcal{M}^2 \rightarrow Gr_{n+1}^W \mathcal{M}^3)$ by the construction in Subsection 6.5. \square

Remark 6.6.5. It is not obvious that the direct sum $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n$ is a splitting of \mathbb{R} -mixed hodge structures for $n \geq 3$ since the map \mathcal{I} does not satisfy $\mathcal{I}(\mathcal{V}_n) \subset \bigoplus_{P+Q=n} \mathcal{V}_{P,Q}$. Our \mathbb{R} -mixed hodge structure is depend on the choice of the map \mathcal{I} .

We consider the pro-nilpotent Lie algebra \mathfrak{u} which is dual to the 1-minimal model \mathcal{M}^* . Then, for $\mathfrak{u} = \bigoplus \mathcal{V}_i^*$, the Lie bracket is dual to the differential $d : \mathcal{V}_k \rightarrow \sum_{i+j=k} \mathcal{V}_i \wedge \mathcal{V}_j$. Hence, $\mathfrak{u} = \bigoplus \mathcal{V}_i^*$ is a graded pro-nilpotent Lie algebra so that $\mathfrak{u}_k = \bigoplus_{i \geq k} \mathcal{V}_i^*$ is the k -th term of the lower central series of \mathfrak{u} . By the \mathbb{R} -mixed hodge structure on \mathcal{M}^* , we obtain the \mathbb{R} -mixed hodge structure on \mathfrak{u} which is compatible with the Lie bracket and T -action. Then, we have $W_{-k}(\mathfrak{u}) = \bigoplus_{i \geq k} \mathcal{V}_i^* = \mathfrak{u}_k$ and so this filtration is the natural filtration which is given by the lower central series of \mathfrak{u} .

7. CONSTRUCTING VMHSS

In this section, we assume the same settings and use same notations as in the previous section.

7.1. Mixed Hodge (T, \mathfrak{u}) -modules.

Definition 7.1.1. Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . Then a T -module structure on V is called *mixed Hodge* if:

- In case $N_0 = 0$, the T -action on V preserves (W_*, F^*) , or
- In case N_0 is odd, for the Lie algebra \mathfrak{t} of T with the \mathbb{R} -Hodge structure induced by (V_0, \mathbf{F}_x^*) , the action $\mathfrak{t} \otimes V \rightarrow V$ is a morphism of \mathbb{R} -mixed Hodge structure.

Lemma 7.1.2. *Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . We suppose that V admit a mixed Hodge T -module structure. Then for any irreducible representation V_α of T , we have:*

- In case $N_0 = 0$, $(V_\alpha^* \otimes V)^T$ admits an \mathbb{R} -mixed Hodge structure so that $W_*((V_\alpha^* \otimes V)^T) = (V_\alpha^* \otimes W_*(V))^T$ and $F_*((V_\alpha^* \otimes V_\mathbb{C})^T) = (V_\alpha^* \otimes F^*(V_\mathbb{C}))^T$
- In case N_0 is odd, $(V_\alpha^* \otimes V)^T$ is an \mathbb{R} -mixed Hodge substructure of $V_\alpha^* \otimes V$.

Proof. The first assertion is easy.

We suppose that N_0 is odd. In this case T is connected and hence $(V_\alpha^* \otimes V)^T = (V_\alpha^* \otimes V)^\mathfrak{t}$. By the assumption, each $W_i(V_\alpha^* \otimes V)$ is a \mathfrak{t} -submodule of $V_\alpha^* \otimes V$. Since any \mathfrak{t} -module is completely reductive, it is sufficient to show that each $(Gr_i^W(V_\alpha^* \otimes V))^T$ is an \mathbb{R} -Hodge substructure of $Gr_i^W(V_\alpha^* \otimes V)$. Thus it is sufficient to prove the statement in the case the \mathbb{R} -mixed Hodge structure on V is pure.

Consider the $U(1)$ -actions on V , V_α , \mathfrak{t} associated with the \mathbb{R} -Hodge structures. By the assumptions, the \mathfrak{t} -actions $\mathfrak{t} \otimes V \rightarrow V$ and $\mathfrak{t} \otimes V_\alpha \rightarrow V_\alpha$ are $U(1)$ -equivariant. By this, we can easily prove that $(V_\alpha^* \otimes V)^\mathfrak{t}$ is a $U(1)$ -submodule of $V_\alpha^* \otimes V$. Thus, we can say that $(V_\alpha^* \otimes V)^T$ is an \mathbb{R} -Hodge substructure of $V_\alpha^* \otimes V$. Hence the Lemma follows. \square

We consider the 1-minimal model \mathcal{M}^* .

Definition 7.1.3. Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . Let $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ be a representation. Denote by $\text{Aut}(V_\mathbb{C}, W_*)$ the group of automorphisms preserving the filtration W_* . For $b \in \text{Aut}(V_\mathbb{C}, W_*)$, a pair (Ω, b) is called *mixed Hodge* if:

- $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ is compatible with the weight filtrations W_* .
- $b^{-1}\Omega b : \mathfrak{u}_\mathbb{C} \rightarrow \text{End}(V_\mathbb{C})$ is compatible with the Hodge filtrations F^* and the conjugate filtrations $\overline{F^*}$.

In particular, if $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ is morphism of \mathbb{R} -mixed Hodge structure, then the pair $(\Omega, 1)$ is mixed Hodge.

For a finite-dimensional \mathbb{R} -vector space V with an \mathbb{R} -mixed Hodge structure (W_*, F^*) , $\mathfrak{n} = W_{-1}(\text{End}(V))$ is a nilpotent Lie algebra. By $W_{-1}\mathfrak{u} = \mathfrak{u}$, if $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ is compatible with the weight filtrations W_* , then $\Omega(\mathfrak{u}) \subset \mathfrak{n}$. A representation $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ is identified with an element $\Omega \in \mathcal{M}^1 \otimes \text{End}(V)$ satisfying the Maurer-Cartan equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$. For this identification, if $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ is compatible with the weight filtrations W_* , then $\Omega \in \mathcal{M}^1 \otimes \mathfrak{n}$. For such Ω and $b \in \text{Aut}(V_\mathbb{C}, W_*)$, $b^{-1}\Omega b : \mathfrak{u}_\mathbb{C} \rightarrow \text{End}(V_\mathbb{C})$ is compatible with the Hodge filtrations F^* and the conjugate filtrations $\overline{F^*}$ if and only if $b^{-1}\Omega b$ is of type $(0, 0)$ i.e.

$$b^{-1}\Omega b \in \bigoplus \left(\mathcal{M}_\mathbb{C}^1 \cap \mathcal{I}^{-1} \left((\mathcal{N}^*)^{P, Q} \right) \right) \otimes \text{End}(V)^{-P, -Q}$$

where $\text{End}(V)^{-P, -Q}$ is the bigrading which is corresponding to the \mathbb{R} -mixed-Hodge structure (W_*, F^*) on $\text{End}(V)$ as in Proposition 2.1.4.

Definition 7.1.4. A mixed Hodge (T, \mathfrak{u}) -representation is (V, W_*, F^*, Ω, b) so that:

- (1) V is a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) .
- (2) V is a T -module and it is mixed Hodge as in Definition 7.1.1.
- (3) A pair (Ω, b) is mixed Hodge as in Definition 7.1.3.
- (4) $\Omega : \mathfrak{u} \rightarrow \text{End}(V)$ is T -equivariant
- (5) b commutes with the T -action on V .

7.2. Flat bundles associated with a mixed Hodge (T, \mathfrak{u}) -module. For a mixed Hodge (T, \mathfrak{u}) -representation (V, W_*, F^*, Ω, b) , we consider the following four flat bundles.

- Define the \mathcal{C}^∞ -vector bundle $\mathbf{E} = \bigoplus_\alpha (V_\alpha^* \otimes V)^T \otimes \mathbf{E}_\alpha$ with the flat connection $D = \bigoplus_\alpha D_\alpha$.
 - We can identify $(A^*(M, \mathcal{O}_\rho) \otimes \text{End}(V))^T$ with $A^*(M, \text{End}(\mathbf{E}))$.
 - By the arguments after Definition 7.1.3, we can regard $\Omega \in (\mathcal{M}^1 \otimes \text{End}(V))^T$ satisfying the Maurer-Cartan equation.
 - By the maps $\phi : \mathcal{M}^* \rightarrow A^*(M, \mathcal{O}_\rho)$ and $\varphi \circ \mathcal{I} : \mathcal{M}_\mathbb{C}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$, we obtain the Maurer-Cartan elements $\Omega_\phi = \phi(\Omega) \in A^1(M, \text{End}(\mathbf{E}))$, $\Omega_\varphi = \varphi(I(\Omega)) \in A^1(M, \text{End}(\mathbf{E}_\mathbb{C}))$ and $\Omega'_\varphi = \varphi(I(\Omega')) \in A^1(M, \text{End}(\mathbf{E}_\mathbb{C}))$ where $\Omega' = b^{-1}\Omega b$.
- Define the flat bundle \mathbf{E}_{Ω_ϕ} as the vector bundle \mathbf{E} with the flat connection $D + \Omega_\phi$.
- Define the flat bundle $\mathbf{E}_{\Omega_\varphi}$ as the vector bundle $\mathbf{E}_\mathbb{C}$ with the flat connection $D + \Omega_\varphi$.
- Define the flat bundle $\mathbf{E}_{\Omega'_\varphi}$ as the vector bundle $\mathbf{E}_\mathbb{C}$ with the flat connection $D + \Omega'_\varphi$.

By Lemma 7.1.2, each $(V_\alpha^* \otimes V)^T$ admits an \mathbb{R} -mixed Hodge structure. Each \mathbf{E}_α is an \mathbb{R} -VHS by $\mathbf{E}_\alpha = S_\lambda \mathbf{E}_0 \cap \mathbf{E}^{[d]}$. Thus, we obtain the increasing filtration \mathbf{W}_* on \mathbf{E} induced by the weight filtrations of $(V_\alpha^* \otimes V)^T$ and the weights of \mathbf{E}_α and decreasing filtration \mathbf{F}^* on $\mathbf{E}_\mathbb{C}$ induced by Hodge filtrations on $(V_\alpha^* \otimes V)^T$ and \mathbf{E}_α .

Lemma 7.2.1. *On any one of the flat bundles \mathbf{E} , \mathbf{E}_{Ω_ϕ} , $\mathbf{E}_{\Omega_\varphi}$ and $\mathbf{E}_{\Omega'_\varphi}$, the filtration \mathbf{W}_* is a filtration of flat bundles. Moreover, for any i , the identity map on \mathbf{E} induces isomorphisms of flat bundles $Gr_i^{\mathbf{W}} \mathbf{E} \cong Gr_i^{\mathbf{W}} \mathbf{E}_{\Omega_\phi}$ and $Gr_i^{\mathbf{W}} \mathbf{E}_\mathbb{C} \cong Gr_i^{\mathbf{W}} \mathbf{E}_{\Omega_\varphi} \cong Gr_i^{\mathbf{W}} \mathbf{E}_{\Omega'_\varphi}$.*

Proof. On \mathbf{E} , the first assertion is obvious.

By the arguments after Definition 7.1.3, we have $\Omega \in (\mathcal{M}^1 \otimes \mathfrak{n})^T$ where $\mathfrak{n} = W_{-1}(\text{End}(V))$. By this, we can say that $\Omega_\phi \wedge \mathbf{W}_i \subset A^1(M, \mathbf{W}_{i-1})$, $\Omega_\varphi \wedge \mathbf{W}_i \subset A^1(M, \mathbf{W}_{i-1})$ and $\Omega'_\varphi \wedge \mathbf{W}_i \subset A^1(M, \mathbf{W}_{i-1})$. This implies the lemma. \square

Proposition 7.2.2. (cf. [5, Lemma 3.11])

$$(D + \Omega'_\varphi)^{1,0} \mathbf{F}^r \subset A^{1,0}(M, \mathbf{F}^{r-1})$$

and

$$(D + \Omega'_\varphi)^{0,1} \mathbf{F}^r \subset A^{0,1}(M, \mathbf{F}^r).$$

Thus the filtration \mathbf{F}^* is a filtration on the holomorphic vector bundle $\mathbf{E}_{\Omega'_\varphi}$ and the Griffiths transversality holds.

Proof. It is sufficient to show

$$(\Omega'_\varphi)^{1,0} \wedge \mathbf{F}^r \subset A^{1,0}(M, \mathbf{F}^{r-1}) \quad \text{and} \quad (\Omega'_\varphi)^{0,1} \wedge \mathbf{F}^r \subset A^{0,1}(M, \mathbf{F}^r).$$

By the bigradings of the \mathbb{R} -mixed Hodge structures on $(V_\alpha^* \otimes V)^T$ and the \mathbb{R} -VHSs \mathbf{E}_α , we have the bigrading $\mathbf{E}_\mathbb{C} = \bigoplus \mathbf{E}^{P,Q}$ of the \mathcal{C}^∞ -vector bundle $\mathbf{E}_\mathbb{C}$. Then we have $\mathbf{F}^r = \bigoplus_{P \geq r} \mathbf{E}^{P,Q}$. By this bigrading, we have the bigrading $\text{End}(\mathbf{E}_\mathbb{C}) = \bigoplus \text{End}(\mathbf{E})^{P,Q}$. We consider the bigrading $A^1(M, \text{End}(\mathbf{E}_\mathbb{C})) = \bigoplus A^1(M, \text{End}(\mathbf{E}))^{P,Q}$ so that

$$A^1(M, \text{End}(\mathbf{E}_\mathbb{C}))^{P,Q} = A^{1,0}(M, \text{End}(\mathbf{E})^{P-1,Q}) \bigoplus A^{0,1}(M, \text{End}(\mathbf{E})^{P,Q-1}).$$

Write $\Omega' = \omega'_1 + \dots + \omega'_l$ such that $\omega'_k \in \left(\bigoplus_{P+Q=k} \mathcal{I}^{-1}((\mathcal{N}^*)^{P,Q}) \otimes \mathfrak{n}_\mathbb{C} \right)^T$. Then we have $d\omega'_k = -\frac{1}{2} \sum_{i+j=k} [\omega'_i, \omega'_j]$. Thus we obtain

$$\varphi(\mathcal{I}(\omega'_k)) = -\frac{1}{2} \sum_{i+j=k} D'F'_g[\varphi(\mathcal{I}(\omega'_i)), \varphi(\mathcal{I}(\omega'_j))].$$

Since Ω' is of type $(0,0)$, $\omega'_1 \in \left(\bigoplus_{P+Q=1} \mathcal{I}^{-1}((\mathcal{N}^*)^{P,Q}) \otimes \mathfrak{n}_\mathbb{C} \right)^T$ is also of type $(0,0)$ and so

$$\varphi(\mathcal{I}(\omega'_1)) \in A^1(M, \text{End}(\mathbf{E}_\mathbb{C}))^{0,0}.$$

Since the map F'_g is of type $(-1, -1)$, inductively, we can say that $\varphi(\mathcal{I}(\omega'_k)) \in A^1(M, \text{End}(\mathbf{E}_\mathbb{C}))^{0, -k+1}$. Thus we have $\Omega'_\varphi \in \bigoplus_{k=1}^l A^1(M, \text{End}(\mathbf{E}_\mathbb{C}))^{0, -k+1}$. This implies

$$(\Omega'_\varphi)^{1,0} \wedge \mathbf{F}^r \subset A^{1,0}(M, \mathbf{F}^{r-1}) \quad \text{and} \quad (\Omega'_\varphi)^{0,1} \wedge \mathbf{F}^r \subset A^{0,1}(M, \mathbf{F}^r).$$

□

Proposition 7.2.3. *There exists a weight preserving gauge transformation a of $\mathbf{E}_\mathbb{C}$ so that:*

- $a(\Omega_\phi) = \Omega_\varphi$ where $a(\Omega_\phi) = a^{-1}da + a^{-1}\Omega_\phi a$.
- a induces the identity map on each $\text{Gr}_i^{\mathbf{W}} \mathbf{E}$.

Moreover, such transformation a can be determined by $H(\Omega) \in A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ where $H : \mathcal{M}_\mathbb{C}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ is a T -equivariant homotopy $H : \mathcal{M}_\mathbb{C}^* \rightarrow A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes [t, dt]$ as in Proposition 6.6.1.

Proof. In [15, Section 5], it is shown that on a nilpotent DGLA (differential graded Lie algebra) L^* , for two Maurer-Cartan elements $x, y \in L^*$, the following two conditions are equivalent:

- x and y are homotopy equivalent i.e. there exists Maurer-Cartan element $x(t) \in L^* \otimes [t, dt]$ so that $x(0) = x$ and $x(1) = y$.
- x and y are gauge equivalent i.e. there exists $A \in L^0$ so that $y = \exp(A) * x$ (see Subsection 9.2).

Consider the DGLA $(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes \mathfrak{n}_\mathbb{C})^T$. We can say the homotopy equivalence of Ω_ϕ and Ω_φ by $H(\Omega)$. Hence we obtain $A \in (A^0(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes \mathfrak{n}_\mathbb{C})^T$ so that $\exp(A)(\Omega_\phi) = \Omega_\varphi$. By $\mathfrak{n} = W_{-1}(\text{End}(V))$, regarding $A \in \text{End}(\mathbf{E}_\mathbb{C})$, we have $A(\mathbf{W}_i) \subset \mathbf{W}_{i-1}$. Thus $a = \exp(A)$ is a desired gauge transformation.

By [15, Lemma 5.6, Proposition 5.7 and the proof of Theorem 5.5], such A is given by $A(1) = A$ for the unique solution $A(t)$ of certain ordinary differential equation associated with the Maurer-Cartan equation of $H(\Omega)$. \square

Since b commutes with the T -action, by Schur's Lemma we have

$$b = \sum_{\alpha} b_{\alpha} \otimes \text{id}_{V_{\alpha}}$$

such that b_{α} is a automorphism of $(V_{\alpha}^* \otimes V)^T$ which preserves the weight filtration. We identify $\text{End}(V)^T$ with the flat sections of $\text{End}(\mathbf{E}_\mathbb{C})$ and we regard b as an automorphism of the flat bundle $\mathbf{E}_\mathbb{C}$. Then we can write $\Omega'_\varphi = b^{-1}\Omega_\varphi b$.

7.3. Main construction. For a mixed Hodge (T, \mathfrak{u}) -representation $\mathfrak{V} = (V, W_*, F^*, \Omega, b)$, we construct an \mathbb{R} -VMHS $(\mathbf{E}_\mathfrak{V}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$. We take:

- $\mathbf{E}_\mathfrak{V} = \mathbf{E}_{\Omega_\phi}$.
- $\mathbf{W}_{\mathfrak{V}*}$ is the increasing filtration \mathbf{W} on the \mathbb{C}^∞ -vector bundle \mathbf{E} .
- $\mathbf{F}_{\mathfrak{V}}^* = ab\mathbf{F}^*$ where a is a weight preserving gauge transformation as in Proposition 7.2.3 and we regard b as an automorphism of the flat bundle $\mathbf{E}_\mathbb{C}$.

Theorem 7.3.1. $(\mathbf{E}_\mathfrak{V}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ is an \mathbb{R} -VMHS.

Proof. By Lemma 7.2.1, $\mathbf{W}_{\mathfrak{V}*}$ is a filtration of the local system $\mathbf{E}_\mathfrak{V}$. By Proposition 7.2.2, $\mathbf{F}_{\mathfrak{V}}^*$ is a filtration of the holomorphic vector bundle $\mathbf{E}_\mathfrak{V} \otimes \mathcal{O}_M$ and the Griffiths transversality holds. We show that $Gr_k^{\mathbf{W}}(\mathbf{E}_\mathfrak{V})$ with the filtration induced by $\mathbf{F}_{\mathfrak{V}}^*$ is an \mathbb{R} -VHS of weight k . We notice that $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ is an \mathbb{R} -VHS. By Lemma 7.2.1, as a local system, we have $Gr_k^{\mathbf{W}}(\mathbf{E}_\mathfrak{V}) = Gr_k^{\mathbf{W}}(\mathbf{E})$. By Proposition 7.2.3 and $b = \sum_{\alpha} b_{\alpha} \otimes \text{id}_{V_{\alpha}}$, $\mathbf{F}_{\mathfrak{V}}^* = ab\mathbf{F}^*$ induces \mathbb{R} -VHS on $Gr_k^{\mathbf{W}}(\mathbf{E}_\mathfrak{V})$. Hence the theorem follows. \square

In general, it is difficult to write a gauge transformation a as in Proposition 7.2.3 explicitly.

Example 7.3.2. We assume that the weight filtration W_* on V is of length 2 i.e. for some k , $W_{k-2}(V) = 0$ and $W_k(V) = V$. Then, by $W_{-3}(\text{End}(V)) = 0$, \mathfrak{n} is 2-step i.e. $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$ and we have $\Omega(W_{-3}(\mathfrak{u})) = 0$. By this, we have $\Omega = \omega_1 + \omega_2$ such that $\omega_1 \in \mathcal{V}_1 \otimes \mathfrak{n}$ and $\omega_2 \in \mathcal{V}_2 \otimes W_{-2}(\text{End}(V))$. By the Maurer-Cartan equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$, we have $d\omega_2 + \frac{1}{2}[\omega_1, \omega_1] = 0$. By constructions of the maps ϕ and φ , we have $\Omega_\phi = \omega_1 + D^c A$ and $\Omega_\varphi = \omega_1 - 2\sqrt{-1}D'A$ where $A = -\frac{1}{2}F_g[\omega_1, \omega_1]$. In this case, a gauge transformation as in Proposition 7.2.3 is $a = \exp(\sqrt{-1}A)$.

Remark 7.3.3. Replace $A^*(M, \mathcal{O}_\rho)$, $(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P, Q}, D', D'')$ and T by the usual de Rham complex $A^*(M)$, usual Dolbeault complex $(A^{*,*}(M), \partial, \bar{\partial})$ and the trivial group respectively. By the arguments as in Section 6 and 7, we obtain Theorem(Prototype). In this case, each \mathbb{R} -VHS on $Gr_k^{\mathbf{W}}(\mathbf{E}_\mathfrak{V})$ is constant and so the

\mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ is unipotent in the sense of Hain-Zucker as in [12]. In this case, we can avoid the argument 6.3 and so we do not need a base point.

8. CONSTRUCTING VMHSs BY SUBSTRUCTURES OF 1-MINIMAL MODELS

8.1. Representations of nilpotent Lie algebras. The reference of this subsection is [18].

Let \mathfrak{n} be a nilpotent Lie algebra and $\mathfrak{n} = \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \mathfrak{n}_3 \dots$ the lower central series of \mathfrak{n} (i.e. $[\mathfrak{n}, \mathfrak{n}_i] = \mathfrak{n}_{i+1}$). It is known that $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$. We say that \mathfrak{n} is k -step if $\mathfrak{n}_k \neq 0$ and $\mathfrak{n}_{k+1} = 0$. Define the increasing filtration W_* of \mathfrak{n} so that $W_{-k} = \mathfrak{n}_k$ for $k > 0$.

Let $U(\mathfrak{n})$ be the universal enveloping algebra. That is $U(\mathfrak{n}) = T(\mathfrak{n})/I$ where $T(\mathfrak{n})$ is the tensor algebra of \mathfrak{n} and I is the ideal which is generated by

$$\{X \otimes Y - Y \otimes X - [X, Y] | X, Y \in \mathfrak{n}\}.$$

Then we have the natural increasing filtration W_* of $U(\mathfrak{n})$ induced by the above increasing filtration of \mathfrak{n} . This filtration is compatible with the multiplication of $U(\mathfrak{n})$. We suppose \mathfrak{n} is k -step. Let $J = W_{-k}(U(\mathfrak{n}))$. Then J is an ideal and the map $\mathfrak{n} \rightarrow U(\mathfrak{n})/J$ induced by the natural inclusion $\mathfrak{n} \hookrightarrow U(\mathfrak{n})$ is an injection. Thus, for the endomorphisms $\text{End}(U(\mathfrak{n})/J)$ of the finite-dimensional vector space $U(\mathfrak{n})/J$, we have a finite-dimensional faithful representation $\tau : \mathfrak{n} \rightarrow \text{End}(U(\mathfrak{n})/J)$. For any $x \in \mathfrak{n}$ and any integer i ,

$$\tau(x)(W_{-i}(U(\mathfrak{n})/J)) \subset W_{-i-1}(U(\mathfrak{n})/J)$$

and so τ is a nilpotent representation.

8.2. Sub-structures of 1-minimal models.

Definition 8.2.1. A k -step sub-structure of \mathcal{M}^* is a sub-vector space $\mathcal{X} \subset \mathcal{M}^1$ so that

- $\mathcal{X} = \mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_k$ such that for each $1 \leq i \leq k$, $\mathcal{X}_i \neq 0$ and $\mathcal{X}_i \subset \mathcal{V}_i$.
- $d : \mathcal{X}_r \rightarrow \sum_{i+j=r} \mathcal{X}_i \wedge \mathcal{X}_j$ (thus $\bigwedge \mathcal{X}$ is a sub-DGA of \mathcal{M}^*).
- \mathcal{X} is a \mathbb{R} -mixed hodge substructure of \mathcal{M}^1 .
- \mathcal{X} is a T -submodule of \mathcal{M}^1 .

We assume that a k -step sub-structure \mathcal{X} is finite dimensional. Consider the nilpotent Lie algebra \mathcal{X}^* which is dual to the DGA $\bigwedge \mathcal{X}$. Take the dual \mathbb{R} -mixed Hodge structure (W_*, F^*) on \mathcal{X}^* . Then, for $\mathcal{X}^* = \mathcal{X}_1^* \oplus \dots \oplus \mathcal{X}_k^*$, the bracket on \mathcal{X}^* is dual to the differential $d : \mathcal{X}_r \rightarrow \sum_{i+j=r} \mathcal{X}_i \wedge \mathcal{X}_j$. Hence, we can easily check that $\mathcal{X}^* = \mathcal{X}_1^* \oplus \dots \oplus \mathcal{X}_k^*$ is a graded nilpotent Lie algebra so that the filtration $W_{-n}(\mathcal{X}^*) = \bigoplus_{i \geq n} \mathcal{X}_i^*$ is the natural filtration which is given by the lower central series of \mathcal{X}^* .

Consider the universal enveloping algebra $U(\mathcal{X}^*)$ of the nilpotent Lie algebra \mathcal{X}^* . Then, we obtain the \mathbb{R} -mixed Hodge structure (W_*, F^*) on $U(\mathcal{X}^*)$ which is induced by the \mathbb{R} -mixed Hodge structure on \mathcal{X}^* . By the above argument, the weight filtration W_* is the natural filtration induced by the lower central series of \mathcal{X}^* . Hence, for

$J = W_{-k}(U(\mathcal{X}))$, we obtain the \mathbb{R} -mixed Hodge structure (W_*, F^*) on the quotient space $U(\mathcal{X}^*)/J$.

Consider the faithful representation $\mathcal{X}^* \rightarrow \text{End}(U(\mathcal{X}^*)/J)$ as in Subsection 8.1. Since the multiplication on $U(\mathcal{X}^*)$ is a morphism of \mathbb{R} -mixed Hodge structure, we can easily show that the representation $\mathcal{X}^* \rightarrow \text{End}(U(\mathcal{X}^*)/J)$ is a morphism of \mathbb{R} -mixed Hodge structure. Consider the composition $\Omega : \mathfrak{u} \rightarrow \text{End}(U(\mathcal{X}^*)/J)$ of the surjection $\mathfrak{u} \rightarrow \mathcal{X}^*$ which is dual to the inclusion $\mathcal{X} \subset \mathcal{M}^1$ and the faithful representation $\mathcal{X}^* \rightarrow \text{End}(U(\mathcal{X}^*)/J)$. Since \mathcal{X} is a T -submodule of \mathcal{M}^1 , we can say that

$$(U(\mathcal{X}^*)/J, W_*, F^*, \Omega, 1)$$

is a mixed Hodge (T, \mathfrak{u}) -representation. Hence, we obtain an \mathbb{R} -VMHS. More precisely, taking a basis x_{i1}, \dots, x_{il_i} of each \mathcal{X}_i and the dual basis $\chi_{i1}, \dots, \chi_{il_i}$ of \mathcal{X}_i^* , we have $\Omega = \sum_{ij} x_{ij} \otimes \chi_{ij}$ and so we can write $\Omega_\phi = \sum_{ij} \phi(x_{ij}) \otimes \chi_{ij}$.

Remark 8.2.2. Each $(\mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_n)$ is a n -step sub-structure of \mathcal{M}^* as in Definition 8.2.1. However, it is not finite-dimensional in general. By the construction, if \mathcal{V}_1 is finite-dimensional, then all \mathcal{V}_i are also finite-dimensional.

Since we have

$$\mathcal{V}_1 \cong H^1(A^*(M, \mathcal{O}_\rho)) \cong \bigoplus_{\alpha} H^1(M, \mathbf{E}_\alpha^*) \otimes V_\alpha,$$

if the group cohomology

$$\bigoplus_{\alpha} H^1(\pi_1(M, x), V_\alpha^*) \otimes V_\alpha.$$

is finite-dimensional, then \mathcal{V}_1 is finite-dimensional.

Proposition 8.2.3. *We suppose the following conditions:*

- $\text{im } \rho$ is a co-compact discrete subgroup in T .
- $T \not\cong O(m, 1)$ for any m .
- The group cohomology $H^1(\ker \rho, \mathbb{R})$ is finite-dimensional.

Then the group cohomology

$$\bigoplus_{\alpha} H^1(\pi_1(M, x), V_\alpha^*) \otimes V_\alpha$$

is finite-dimensional. Hence, in this case, $(\mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_n)$ is a finite-dimensional n -step sub-structure of \mathcal{M}^ and so we obtain the \mathbb{R} -VMHS associated with $(\mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_n)$ for each n .*

Proof. Let T_0 be the identity component of T and $\Gamma = \text{im } \rho \cap T_0$. Then Γ is a co-compact discrete subgroup of the connected semi-simple Lie group T_0 . We notice that Γ is a finite-index normal subgroup of $\text{im } \rho$. For the extension

$$1 \longrightarrow \ker \rho \longrightarrow \pi_1(M, x) \longrightarrow \text{im } \rho \longrightarrow 1,$$

we have the spectral sequence $E_r^{p,q}$ so that

$$E_2^{p,q} = \bigoplus_{\alpha} H^p(\text{im } \rho, H^q(\ker \rho, \mathbb{R}) \otimes V_\alpha^*) \otimes V_\alpha$$

and it converges to $\bigoplus_{\alpha} H^{p+q}(\pi_1(M, x), V_{\alpha}^*) \otimes V_{\alpha}$. It is sufficient to show that $E_2^{1,0}$ and $E_2^{0,1}$ are finite-dimensional. By Raghunathan's result in [19], for non-trivial V_{α} , we have $H^1(\Gamma, V_{\alpha}) = 0$. Since Γ is a finite-index normal subgroup of $\text{im}\rho$, we have $H^1(\text{im}\rho, V_{\alpha}) = 0$. Thus

$$E_2^{1,0} = \bigoplus_{\alpha} H^1(\text{im}\rho, H^0(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*) \otimes V_{\alpha}$$

is finite-dimensional. We have

$$E_2^{0,1} = \bigoplus_{\alpha} H^0(\text{im}\rho, H^1(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*) \otimes V_{\alpha} = \bigoplus_{\alpha} (H^1(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*)^{\text{im}\rho} \otimes V_{\alpha}.$$

Since $\text{im}\rho$ is Zariski-dense in T , we have

$$\bigoplus_{\alpha} (H^1(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*)^{\text{im}\rho} \otimes V_{\alpha} = \bigoplus_{\alpha} (H^1(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*)^T \otimes V_{\alpha} \cong H^1(\ker\rho, \mathbb{R}).$$

Since $H^1(\ker\rho, \mathbb{R})$ is finite-dimensional, we can say that $E_2^{0,1}$ is finite-dimensional. Hence the proposition follows. \square

8.3. Lower step \mathbb{R} -VMHS. By using sub-structures of 1-minimal models, we can obtain explicit \mathbb{R} -VMHSs whose weight filtration is of length 1 or 2.

8.3.1. 1-step \mathbb{R} -VMHS. Let $\{V_{\alpha_1}, \dots, V_{\alpha_l}\}$ be a finite set of irreducible representations of T . Take $\mathcal{X}_1 = \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i}) \otimes V_{\alpha_i}^*$. Then, regarding \mathcal{X}_1 as a subspace in \mathcal{V}_1 , obviously \mathcal{X}_1 is a finite-dimensional 1-step sub-structure of \mathcal{M}^* . In this case we have $U(\mathcal{X}_1^*)/J = \langle 1 \rangle \oplus \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^* \otimes V_{\alpha_i}$. Take a basis $x_1^{\alpha_i}, \dots, x_{m_i}^{\alpha_i}$ of each $\mathcal{H}^1(M, \mathbf{E}_{\alpha_i}^*)$. We take the dual basis $\chi_1^{\alpha_i}, \dots, \chi_{m_i}^{\alpha_i}$ of each $\mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^*$. Then, the \mathbb{R} -VHMS as in subsection 8.2 is given by the flat connection $D + \sum x_j^{\alpha_i} \otimes \chi_j^{\alpha_i}$ over the vector bundle $\mathbb{R} \oplus \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i}$. Precisely, for $(f, \sum \eta_{\alpha_i}) \in \mathbb{R} \oplus \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i}$, we have

$$\left(D + \sum x_j^{\alpha_i} \otimes \chi_j^{\alpha_i} \right) \left(f, \sum \eta_{\alpha_i} \right) = \left(df, \sum D_{\alpha_i} \eta_{\alpha_i} + f \sum x_j^{\alpha_i} \otimes \chi_j^{\alpha_i} \right)$$

The weight filtration \mathbf{W}_* is given by

$$\mathbf{W}_0 = \mathbb{R} \oplus \bigoplus_i H^1(M, \mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i},$$

$$\mathbf{W}_{-1} = \bigoplus_i H^1(M, \mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i}$$

and $\mathbf{W}_{-2} = 0$. The Hodge filtration \mathbf{F}^* on $(\mathbb{R} \oplus \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i}) \otimes \mathbb{C}$ is given by the dual Hodge structures on $\mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^*$ and the \mathbb{R} -VHSs \mathbf{E}_{α_i} . Similar construction is given in [4, Example 2.2.2].

8.3.2. *2-step \mathbb{R} -VMHS.* Take $\mathcal{X}_1 = \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i}) \otimes V_{\alpha_i}^*$ as the above argument. Let $\mathcal{X}_2 = d^{-1}(\mathcal{X}_1 \wedge \mathcal{X}_1) \subset \mathcal{V}_2$. As in Remark 6.6.2, we have $\mathcal{I}(\mathcal{X}_2 \otimes \mathbb{C}) = d^{-1}(\mathcal{I}(\mathcal{X}_1 \otimes \mathbb{C}) \wedge \mathcal{I}(\mathcal{X}_1 \otimes \mathbb{C}))$. Hence, $\mathcal{X}_1 \oplus \mathcal{X}_2$ is a finite-dimensional sub-structure of \mathcal{M}^* and so we can construct the \mathbb{R} -VMHS associated with $\mathcal{X}_1 \oplus \mathcal{X}_2$. Thus, we obtain the \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ associated with the mixed Hodge (T, \mathbf{u}) -module $\mathfrak{V} = (U(\mathcal{X}^*)/J, W_*, F^*, \Omega, 1)$. Since the weight filtration W_* is of length 2, we can write the Hodge filtration $\mathbf{F}_{\mathfrak{V}}^*$ as in Example 7.3.2.

For this construction, it is necessary that the kernel of the cup product

$$\left(\bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i}) \otimes V_{\alpha_i}^* \right) \wedge \left(\bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i}) \otimes V_{\alpha_i}^* \right) \rightarrow H^2(M, \mathcal{O}_{\rho})$$

is non-trivial. On Example 6.3.2, by $\dim M = 2$, we have

$$H^2(M, \mathcal{O}_{\rho}) = \bigoplus_{k=0}^{\infty} H^2(M, S^k \mathbf{E}_0^*) \otimes S^k V_0 \cong \mathbb{R}.$$

By the Euler number $\chi(M) = 2 - 2g$ and $\dim V_0 = 2$, we have $\dim H^1(M, \mathbf{E}_0^*) \otimes V_0 = 8g - 8 \geq 8$. Thus the kernel of the cup product

$$(\mathcal{H}^1(M, \mathbf{E}_0^*) \otimes V_0) \wedge (\mathcal{H}^1(M, \mathbf{E}_0^*) \otimes V_0) \rightarrow H^2(M, \mathcal{O}_{\rho})$$

is non-trivial.

9. CONSTRUCTING \mathbb{R} -VMHS FROM DEFORMATION THEORY

9.1. **Functors of Artinian algebras.** We briefly review the theory of Schlessinger's hull ([20]). Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . For a local \mathbb{K} -algebra R , we denote by \mathfrak{m}_R the maximal ideal of R . We define:

- Art is the category of Artinian local \mathbb{K} -algebras.
- $\overline{\mathit{Art}}$ is the category of complete Noetherian local \mathbb{K} -algebra R so that $R/\mathfrak{m}_R^k \in \mathit{Art}$ for any k .
- A functor F of Art is a covariant functor from Art to the category of sets so that $F(\mathbb{K})$ is a 1-point set.
- For a functor F of Art , we define $t_F = F(\mathbb{K}[t]/(t^2))$.
- For $R \in \overline{\mathit{Art}}$, we define the functor h_R of Art as $h_R(A) = \text{Hom}(R, A)$.
- For a functor F of Art , $R \in \overline{\mathit{Art}}$ and $\xi \in F(R)$, we define the morphism $T_{\xi} : h_R \rightarrow F$ of functors such that $h_R(A) \ni u \mapsto F(u)(\xi) \in F(A)$ where we extend F to $\overline{\mathit{Art}}$.
- A morphism $F \rightarrow G$ of functors of Art is *smooth* if for any surjection $B \rightarrow A$ in Art the map

$$F(B) \rightarrow F_1(A) \times_{F_2(A)} F_2(B)$$

is surjective.

- A morphism $F \rightarrow G$ of functors of Art is *étale* if it is smooth and the induced map $t_F \rightarrow t_G$ is bijective.
- For a functor F of Art , $R \in \overline{\mathit{Art}}$ and $\xi \in F(R)$, a pair (R, ξ) is a *hull* if the morphism $T_{\xi} : h_R \rightarrow F$ is étale.

The following uniqueness is important.

Proposition 9.1.1 ([20]). *For a functor F of \mathbf{Art} , if two pairs (R_1, ξ_1) and (R_2, ξ_2) are hulls, then we have an isomorphism $u : R_1 \rightarrow R_2$ such that $F(u)(\xi_1) = \xi_2$.*

9.2. Deformation theory of DGLA. We briefly review the Kuranishi theory of a DGLA (differential graded Lie algebra) ([15]). Let L^* be a DGLA over \mathbb{K} with a differential d . We define $MC(L^*) = \{x \in L^1 \mid dx + \frac{1}{2}[x, x] = 0\}$. If L^* is nilpotent, then for any $B \in L^0$, we can define the gauge transformation $\exp(B)^*$ on L^1 by the exponential of the affine transformation $x \mapsto [B, x] - dB$. Moreover this action preserves $MC(L)$ and so we can define gauge equivalence on $MC(L)$ so that $x, y \in MC(L)$ are gauge equivalent if for some $B \in L^0$, $y = \exp(B)^*(x)$.

For any DGLA L^* and $A \in \mathbf{Art}$, the DGLA $L^* \otimes \mathfrak{m}_A$ is nilpotent and we define $Def_L(A)$ as the set of gauge equivalent classes of $MC(L^* \otimes \mathfrak{m}_A)$. Consider the functor $Def_L : A \mapsto Def_L(A)$ of \mathbf{Art} . We notice that for $R \in \overline{\mathbf{Art}}$ we can also define the gauge transformation $\exp(B)^*$ of $B \in L^0 \otimes \mathfrak{m}_S$ by using formal power series and we can regard $Def_L(S)$ as the set of gauge equivalent classes of $MC(L^* \otimes \mathfrak{m}_S)$.

Let L^* be a DGLA such that the cohomologies $H^0(L^*)$, $H^1(L^*)$ and $H^2(L^*)$ are finite-dimensional. We will define the Kuranishi functor of L^* as in [15, Section 4]. For our application, we only consider the special case. A *special grading* is a grading $L^* = \bigoplus_{k \geq 0} L_k^*$ on the vector space L^* so that:

- $dL_k \subset L_k$ and $[L_{k_1}^*, L_{k_2}^*] \subset L_{k_1+k_2}^*$.
- $L_0^* = L^0$.
- $L^1 = \bigoplus_{k \geq 1} L_k^1$ and $\ker d|_{L^1} = L_1^1$.
- $L^2 = \bigoplus_{k \geq 2} L_k^2$.
- $d : L_2^1 \rightarrow \ker d|_{L_2^2}$ is injective and $d : L_k^1 \rightarrow \ker d|_{L_k^2}$ is bijective for any $k \geq 3$.

This grading gives a special case of decomposition as in [15, Section 4].

We assume that L^* admits a special grading. For $A \in \mathbf{Art}$, we define the set

$$Kur_L(A) = \{x_1 \in L_1^1 \otimes \mathfrak{m}_A \mid [x_1, x_1] \equiv 0 \in H^2(L^*) \otimes \mathfrak{m}_A\}$$

and the map $Kur_L(A) \ni x_1 \mapsto \sum_i x_i \in MC(L^* \otimes \mathfrak{m}_A)$ so that for $k \geq 2$,

$$dx_k = -\frac{1}{2} \sum_{i+j=k, i>0, j>0} [x_i, x_j].$$

By the assumptions, each x_k is uniquely determined by x_1 . We obtain the functor $Kur_L : A \mapsto Kur_L(A)$ and the morphism $Kur_L \rightarrow Def_L$ of functors so that $Kur_L(A) \ni x_1 \mapsto [\sum_i x_i] \in Def_L(A)$.

Theorem 9.2.1. ([15, Theorem 4.7]) The morphism $Kur_L \rightarrow Def_L$ is étale.

We can easily check that $Kur_L = h_R$ so that $R = \mathbb{K}[[L_1^1]^*]/I$ where $\mathbb{K}[[L_1^1]^*]$ is the algebra of formal power series on $(L_1^1)^*$ and I is the ideal generated by the quadratic polynomials on $(L_1^1)^*$ associated with $L_1^1 \ni x \mapsto [x, x] \in H^2(L^*)$. Take $\xi_1 \in L^* \otimes R$ which is the extension of the identity map $I \in L_1^1 \otimes (L_1^1)^*$. Then, by elementary arguments, we have $[\xi_1, \xi_1] \equiv 0 \in H^2(L^*) \otimes \mathfrak{m}_R$ (see [5, Lemma 3.4]).

Take the formal power series $\xi = \sum_{i=1}^{\infty} \xi_i \in MC(L^* \otimes \mathfrak{m}_R)$ so that

$$d\xi_k = -\frac{1}{2} \sum_{i+j=k, i>0, j>0} [\xi_i, \xi_j].$$

By Theorem 9.2.1, we have:

Corollary 9.2.2. *For the functor Def_L of Art and the gauge equivalent classe $[\xi] \in Def_L(R)$ of ξ , the pair $(R, [\xi])$ is a hull.*

We notice that the definition of R is independent of the choice of a special grading $L^* = \bigoplus_{k \geq 0} L_k^*$ as above but ξ varies for the choice of a special grading. By Proposition 9.1.1, we have:

Corollary 9.2.3. *For two Maurer-Cartan elements $\xi, \xi' \in MC(L^* \otimes \mathfrak{m}_R)$ constructed as above associated with two special gradings of L^* , there exists an automorphism $u : R \rightarrow R$ of R and $B \in L^0 \otimes \mathfrak{m}_R$ such that*

$$\xi' = \exp(B) * u(\xi) = \exp(\text{ad}_B) \circ u(\xi).$$

9.3. Mixed Hodge (T, u) -module associated with deformation theory. We assume the same settings and use same notations as in Section 6 and 7. Additionally, we assume the weight N_0 of \mathbf{E}_0 is 0. Let U be a finite-dimensional rational T -representation. Then, we have an irreducible decomposition $U = \bigoplus V_{\gamma_i}$. Corresponding to this, we have the \mathbb{R} -VHS $\mathbf{E}_U = \bigoplus \mathbf{E}_{\gamma_i}$.

We consider the DGLA $L^* = (\mathcal{M}^* \otimes \text{End}(U))^T$ over \mathbb{R} . Then we have the grading

$$L^* = \bigoplus_k (\mathcal{M}_k^* \otimes \text{End}(U))^T.$$

This grading is a special grading and $L_1^* = (\mathcal{V}_1 \otimes \text{End}(U))^T = \mathcal{H}^1(M, \text{End}(\mathbf{E}_U))$. We study $R = \mathbb{K}[[L_1^*]]/I$ as in the last subsection. Since the map $\phi : \mathcal{M}^* \rightarrow A^*(M, \mathcal{O}_\rho)$ induces an injection on the second cohomology, the map $L^* \rightarrow (A^*(M, \mathcal{O}_\rho) \otimes \text{End}(U))^T \cong A^*(M, \text{End}(\mathbf{E}_U))$ induces an injection $H^2(L^*) \rightarrow H^2(M, \text{End}(\mathbf{E}_U))$. Thus I is the ideal generated by the quadratic polynomials on $\mathcal{H}^1(M, \text{End}(\mathbf{E}_U))^*$ associated with $\mathcal{H}^1(M, \text{End}(\mathbf{E}_U)) \ni x \mapsto [x, x] \in H^2(M, \text{End}(\mathbf{E}_U))$. Define $I_2 \subset S^2 \mathcal{H}^1(M, \text{End}(\mathbf{E}_U))^*$ by the image of the dual

$$H^2(M, \text{End}(\mathbf{E}_U))^* \rightarrow S^2 \mathcal{H}^1(M, \text{End}(\mathbf{E}_U))^*$$

of the cup bracket

$$[,] : \mathcal{H}^1(M, \text{End}(\mathbf{E}_U)) \times \mathcal{H}^1(M, \text{End}(\mathbf{E}_U)) \rightarrow H^2(M, \text{End}(\mathbf{E}_U)).$$

We have

$$R/\mathfrak{m}_R^k \cong \bigoplus_{i=1}^{k-1} S^i \mathcal{H}^1(M, \text{End}(\mathbf{E}_U))^* / S^{i-2} \mathcal{H}^1(M, \text{End}(\mathbf{E}_U))^* \cdot I_2.$$

It is known that the cup bracket

$$[,] : \mathcal{H}^1(M, \text{End}(\mathbf{E}_U)) \times \mathcal{H}^1(M, \text{End}(\mathbf{E}_U)) \rightarrow H^2(M, \text{End}(\mathbf{E}_U))$$

is a homomorphism of \mathbb{R} -Hodge structure for the \mathbb{R} -Hodge structures on $\mathcal{H}^1(M, \text{End}(\mathbf{E}_U))$ and $H^2(M, \text{End}(\mathbf{E}_U)) \cong \mathcal{H}^2(M, \text{End}(\mathbf{E}_U))$ as in Section 5. By this, we can say that

$$R/\mathfrak{m}_R \leftarrow R/\mathfrak{m}_R^2 \leftarrow R/\mathfrak{m}_R^3 \cdots$$

is a inverse system of split \mathbb{R} -mixed Hodge structures such that $W_i(R/\mathfrak{m}_R^k) = \mathfrak{m}_R^i/\mathfrak{m}_R^k$ and the multiplication on R is compatible with these structures.

Take the formal power series $\xi = \sum_{i=1}^{\infty} \xi_i \in L^* \otimes \mathfrak{m}_R$ associated with the above special grading as in the last subsection. For the complexification $L_{\mathbb{C}}^* = (\mathcal{M}_{\mathbb{C}}^* \otimes \text{End}(U_{\mathbb{C}}))^T$, we have another special grading

$$L_{\mathbb{C}}^* = \bigoplus_k \left(\bigoplus_{P+Q=k} \mathcal{I}^{-1}((\mathcal{N}^*)^{P,Q}) \otimes \text{End}(U) \right)^T.$$

Take the formal power series $\xi' = \sum_{i=1}^{\infty} \xi'_i \in L_{\mathbb{C}}^* \otimes \mathfrak{m}_{R_{\mathbb{C}}}$ associated with this new special grading. Then, by Proposition 9.2.3, there exists an automorphism $u : R \rightarrow R$ of R and $B \in L^0 \otimes \mathfrak{m}_R$ such that

$$\xi' = \exp(\text{ad}_B) \circ u(\xi).$$

We consider the map $\iota : R \rightarrow \text{End}(R)$ associated with the multiplication on R . Let

$$\Omega = \iota(\xi) \in L^* \otimes \text{End}(R) = (\mathcal{M}^* \otimes \text{End}(U))^T \otimes \text{End}(R)$$

and

$$\Omega' = \iota(\xi') \in L_{\mathbb{C}}^* \otimes \text{End}(R_{\mathbb{C}}) = (\mathcal{M}_{\mathbb{C}}^* \otimes \text{End}(U_{\mathbb{C}}))^T \otimes \text{End}(R_{\mathbb{C}}).$$

Then we have

$$\Omega' = b^{-1} \Omega b.$$

where $b = u^{-1} e^{-B}$.

Consider each quotient $q_k : R \rightarrow R/\mathfrak{m}_R^k$. Take $\xi(k) = q_k(\xi)$, $\xi'(k) = q_k(\xi')$, $B_k = q_k(B)$ and the reduction $u_k : R/\mathfrak{m}_R^k \rightarrow R/\mathfrak{m}_R^k$ of $u : R \rightarrow R$. We have $\xi'(k) = \exp(\text{ad}_{B_k}) \circ u_k(\xi(k))$. By the construction, we have $\xi(k) \equiv \sum_{i=1}^{k-1} \xi_i$ and $\xi'(k) \equiv \sum_{i=1}^{k-1} \xi'_i$. For the map $\iota_k : R/\mathfrak{m}_R^k \rightarrow \text{End}(R/\mathfrak{m}_R^k)$ associated with the multiplication on R/\mathfrak{m}_R^k , let $\Omega_k = \iota_k(\xi(k))$ and $\Omega'_k = \iota_k(\xi'(k))$. We have

$$\Omega'_k = b_k^{-1} \Omega_k b_k.$$

where $b_k = u_k^{-1} e^{-B_k}$. Consider the \mathbb{R} -mixed Hodge structure (W_*, F^*) on R/\mathfrak{m}_R^k as above. We regard Ω as a T -equivariant Lie algebra homomorphism $\mathfrak{u} \rightarrow \text{End}(U) \otimes \text{End}(R/\mathfrak{m}_R^k)$. We have:

Proposition 9.3.1. *$(U \otimes R/\mathfrak{m}_R^k, W_*, F^*, \Omega_k, b_k)$ is a mixed Hodge (T, \mathfrak{u}) -representation.*

Proof. By the above arguments, it is sufficient to prove the following two claims

- $\Omega_k : \mathfrak{u} \rightarrow \text{End}(U) \otimes \text{End}(R/\mathfrak{m}_R^k)$ is compatible with the weight filtrations W_* .
- $\Omega'_k : \mathfrak{u}_{\mathbb{C}} \rightarrow \text{End}(U_{\mathbb{C}}) \otimes \text{End}(R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k)$ is compatible with the Hodge filtrations F^* and the conjugate filtrations \bar{F}^* .

Consider the sum $\xi(k) \equiv \sum_{i=1}^{k-1} \xi_i$. By the construction, we have $\xi_i \in \mathcal{V}_k \otimes \mathfrak{m}_{R_{\mathbb{C}}}^k$. This implies the first claim.

Consider the sum $\xi'(k) \equiv \sum_{i=1}^{k-1} \xi'_i$. By the splitting

$$R/\mathfrak{m}_R^k \cong \bigoplus_{i=1}^{k-1} S^i \mathcal{H}^1(M, \text{End}(\mathbf{E}_U))^* / S^{i-2} \mathcal{H}^1(M, \text{End}(\mathbf{E}_U))^* \cdot I_2,$$

we took $\xi'_1 \in$ as a identity map on $\mathcal{H}^1(M, \text{End}(\mathbf{E}_U)) \otimes \mathbb{C}$. Thus for the bigrading of the split \mathbb{R} -mixed hodge structure on R/\mathfrak{m}_R^k and the bigrading $\mathcal{M}_{\mathbb{C}}^* = \mathcal{I}^{-1}(\bigoplus (\mathcal{N}^*)^{P,Q})$, $\xi'_1 \in \mathcal{M}_{\mathbb{C}}^* \otimes \text{End}(U) \otimes R/\mathfrak{m}_R^k$ is of type $(0, 0)$. Since we have $d\xi_l = -\frac{1}{2} \sum_{i+j=l, i>0, j>0} [\xi_i, \xi_j]$ for each l , we can say that each ξ_l is also of type $(0, 0)$. Thus the sum $\xi'(k) \equiv \sum_{i=1}^{k-1} \xi'_i$ is of type $(0, 0)$. This implies the second claim. Thus the proposition follows. \square

Thus we can construct \mathbb{R} -VMHSs associated with mixed Hodge (T, \mathbf{u}) -representations $(U \otimes R/\mathfrak{m}_R^k, W_*, F^*, \Omega_k, b_k)$. The idea of this construction is inspired by Eyssidieux-Simpson's work in [5]. In [5, Theorem 3.15], Eyssidieux and Simpson construct \mathbb{R} -VMHSs starting from an \mathbb{R} -VHS \mathbf{E} , by using Goldman-Millson's theory in [8] and [9]. The construction of this section is very similar to Eyssidieux-Simpson's construction. They also use the mixed Hodge structure on

$$\bigoplus_{i=1}^{k-1} S^i \mathcal{H}^1(M, \text{End}(\mathbf{E}))^* / S^{i-2} \mathcal{H}^1(M, \text{End}(\mathbf{E}))^* \cdot I_2$$

(see [5, Subsection 2.3]). But they do not use 1-minimal model and it is not clear that the two constructions are same. This matter is left for future work.

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(Hisashi Kasuya)

- DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, OSAKA, JAPAN.
- INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE, PARIS, FRANCE

E-mail address: kasuya@math.sci.osaka-u.ac.jp